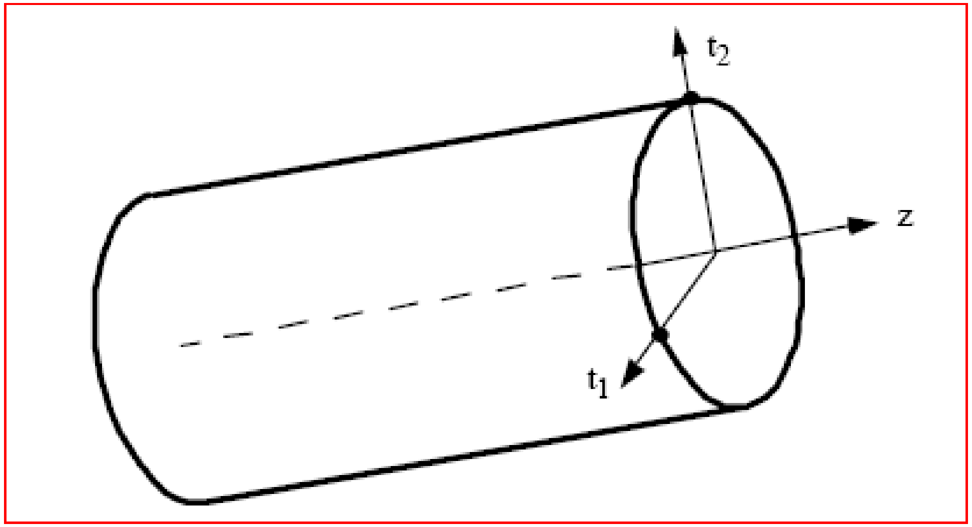
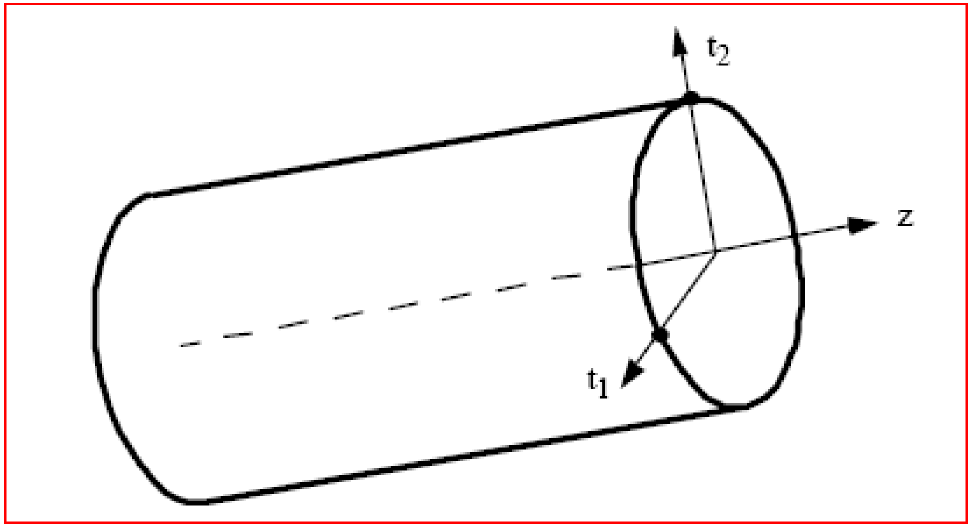
Propagation modes

Guided propagation



We talk about guided propagation for waves travelling along a structure having a invariant geometry in the plane transverse to the propagation direction

Guided propagation



for all vector $\mathbf{v}$ (t_1,t_2,z),
we can write :
$\mathbf{v}\ (t_1,t_2,z)=v_z(t_1,t_2,z)\mathbf{z}+\mathbf{v_t}(t_1,t_2,z)$

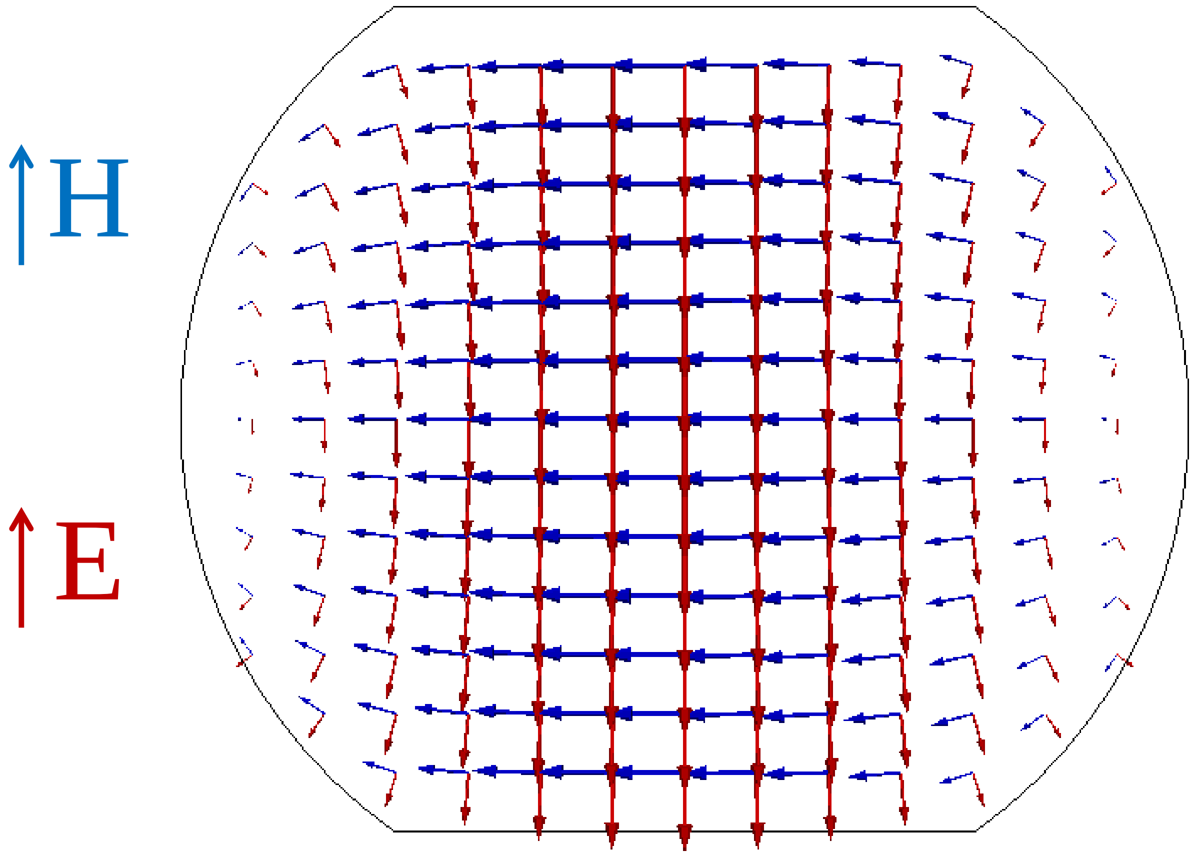
v_z is the longitudinal component, $\mathbf{v_t}$ the transverse component

The vector operators become :

$$\nabla = \frac{\partial}{\partial z}\hat{\mathbf{z}} + \nabla_{\mathbf{t}}$$

$$\nabla^2 = \frac{\partial^2}{\partial z^2} + \nabla_t^2$$

Guided propagation

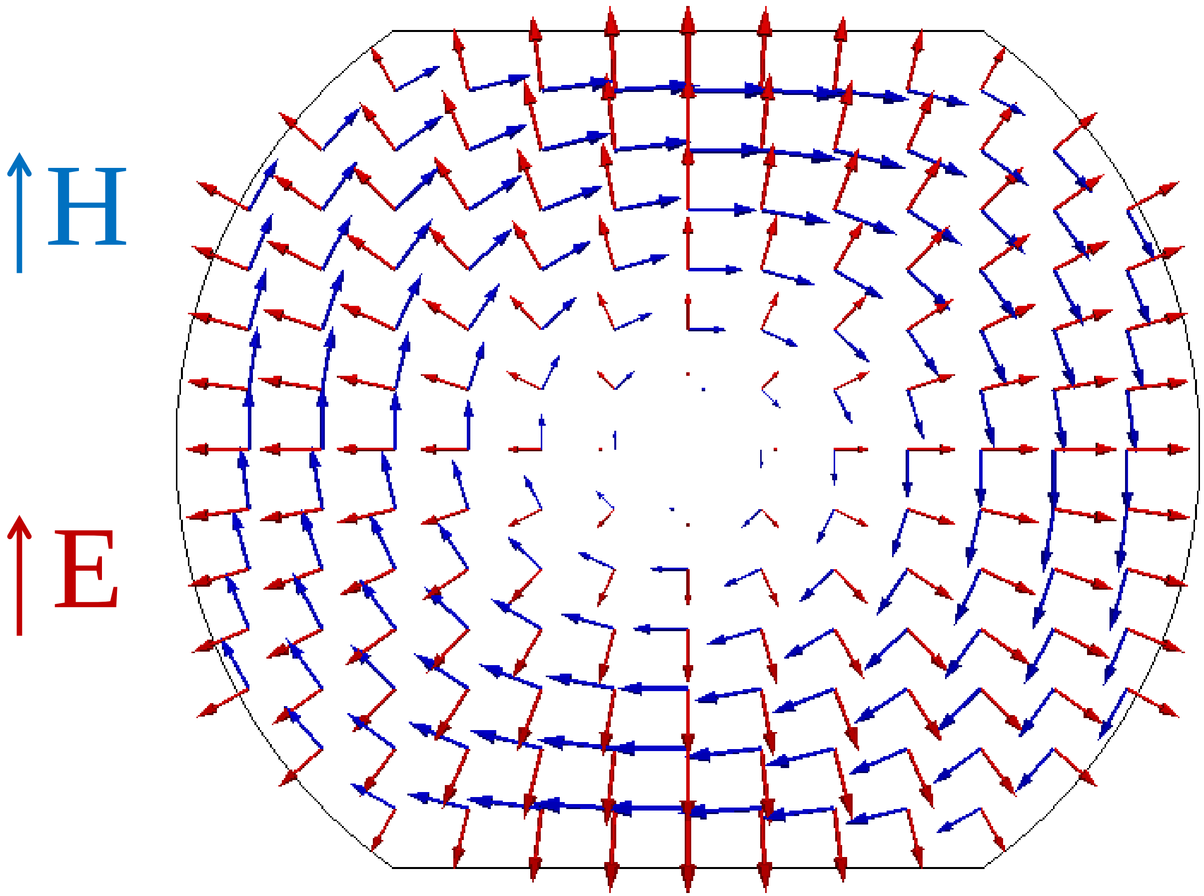


Boundary conditions
For CEP boundaries:

$$\mathbf{E}^{\parallel} = \mathbf{0}$$

$$\mathbf{B}^{\perp} = \mathbf{0}$$

Guided propagation



Boundary conditions:

$$\mathbf{E}^{\parallel} = \mathbf{0}$$

$$\mathbf{B}^{\perp} = \mathbf{0}$$

Guided propagation

All function $f\left(t_1,t_2,z\right)=E_z,E_{t1},E_{t2},H_z,H_{t1},H_{t2}$
will be considered as the product (separation of variables) :

$$f\left(t_1,t_2,z\right)=T\left(t_1,t_2\right)Z\left(z\right)$$

Helmholtz equation becomes :

$$\left(\nabla^2+k^2\right)f=0$$

$$\left(\nabla^2+k^2\right)T\left(t_1,t_2\right)Z\left(z\right)=0$$

$$\nabla_t^{\,2}T\left(t_1,t_2\right)Z\left(z\right)+T\left(t_1,t_2\right)\frac{d^2Z\left(z\right)}{dz^2}+k^2T\left(t_1,t_2\right)Z\left(z\right)=0$$

$$\Rightarrow \frac{\nabla_t^2T}{T}+k^2+\frac{1}{Z}\frac{d^2Z}{dz^2}=0$$

Guided propagation

This relation has to be valid for all (t_1, t_2, z), thus

$$\frac{\nabla_t^2 T}{T} + k^2 = -\gamma^2$$

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = \gamma^2$$

The longitudinal dependency has an easy analytical solution :

$$Z(z) = Ae^{-\gamma z} + Be^{+\gamma z}$$

It is the superposition of an incident wave and a reflected wave

Guided propagation

γ : longitudinal propagation constant

$$\gamma = \alpha + j\beta$$

α : lineic attenuation [neper/m] or [dB/m]
β : lineic phase shift [rad/m]

Guided propagation

If we consider only incident waves, B=0.
We can then write for the electromagnetic fields :

$$\mathbf{E}(t_1, t_2, z) = \mathbf{e}(t_1, t_2) e^{-\gamma z} = (e_z \hat{\mathbf{z}} + \mathbf{e_t}) e^{-\gamma z}$$

$$\mathbf{H}(t_1, t_2, z) = \mathbf{h}(t_1, t_2) e^{-\gamma z} = (h_z \hat{\mathbf{z}} + \mathbf{h_t}) e^{-\gamma z}$$

The transverse part will of course depend on the transverse geometry of the guide

Transverse dependency

$$\left(\nabla_t^2 + k^2 + \gamma^2\right)\mathbf{e} = \left(\nabla_t^2 + k_c^2\right)\mathbf{e} = 0$$

$$\left(\nabla_t^2 + k^2 + \gamma^2\right)\mathbf{h} = \left(\nabla_t^2 + k_c^2\right)\mathbf{h} = 0$$

$$k_c^2 = k^2 + \gamma^2$$

The admissible k_c are the eigenvalues of the guide, the corresponding **e** and **h** the eigenmodes of the guide

longitudinal propagation constant

A propagation constant corresponds to each eigenvalue :

$$\gamma = \sqrt{k_c^2 - k^2} \qquad k = \omega\sqrt{\varepsilon\mu}$$

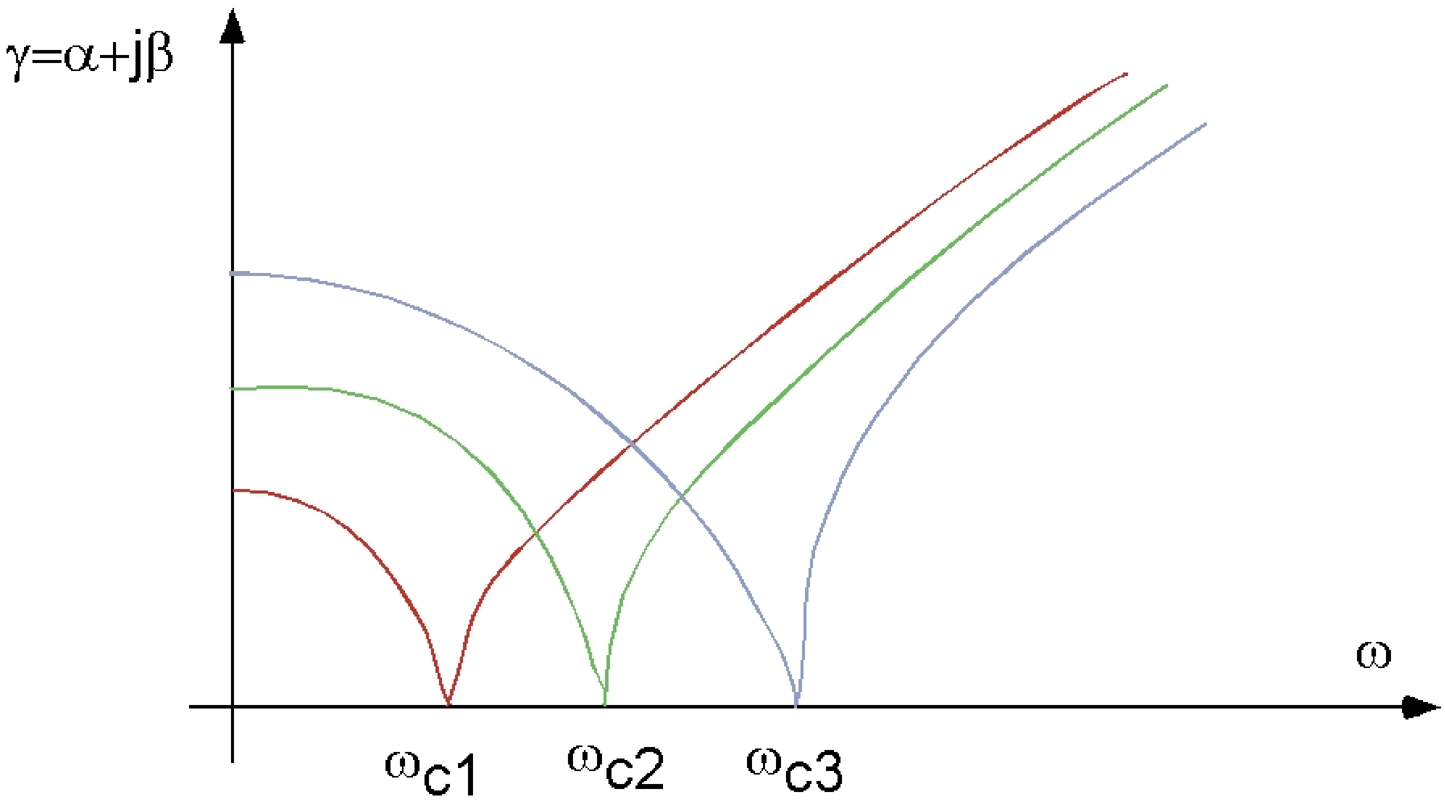
Thus, in a lossless case, depending on the relation between k and k_c the longitudinal propagation constant can be real or imaginary :

$$\text{if } k_c > \omega\sqrt{\varepsilon\mu} \;\; \text{then } \gamma = \alpha$$

$$\text{if } k_c = \omega\sqrt{\varepsilon\mu} \;\; \text{then } \gamma = 0$$

$$\text{if } k_c < \omega\sqrt{\varepsilon\mu} \;\; \text{then } \gamma = j\beta$$

Dispersion diagram



Guided propagation

The 6 components of the fields are not independent, the six components of the modes neither. Indeed, through Maxwell's equations we can write:

$$\nabla_{\mathbf{t}} \times \mathbf{e}_{\mathbf{t}} = -j\omega\mu\left(h_z \hat{\mathbf{z}}\right)$$

$$\nabla_{\mathbf{t}} \times \mathbf{h}_{\mathbf{t}} = j\omega\varepsilon\left(e_z \hat{\mathbf{z}}\right)$$

and we get finally (after some manipulations) :

$$k_c^2 \mathbf{e}_{\mathbf{t}} = -\gamma\nabla_t e_z + j\omega\mu\hat{\mathbf{z}} \times \nabla_t h_z$$

$$k_c^2 \mathbf{h}_{\mathbf{t}} = -\gamma\nabla_t h_z - j\omega\varepsilon\hat{\mathbf{z}} \times \nabla_t e_z$$

Guided propagation

Thus, for a given problem, we need only to compute e_z and h_z. The other components will be obtained using Maxwell's equations

Strategy

1. Compute e_z and h_z solving : $\left(\nabla_t^2 + k_c^2\right)e_z = 0 \;;\; \left(\nabla_t^2 + k_c^2\right)h_z = 0$
2. Compute the longitudinal propagation constant. For an incident wave, select the root with Im(γ)>0
$$\gamma = \alpha + j\beta = \sqrt{k_c^2 - k^2}$$
3. Compute the transverse components of the modes
$$k_c^2\mathbf{e_t} = -\gamma\nabla_t e_z + j\omega\mu\hat{\mathbf{z}}\times\nabla_t h_z$$
$$k_c^2\mathbf{h_t} = -\gamma\nabla_t h_z - j\omega\varepsilon\hat{\mathbf{z}}\times\nabla_t e_z$$
4. Reconstruct the electromagnetic fields
$$\mathbf{E}(t_1, t_2, z) = \left(e_z\hat{\mathbf{z}} + \mathbf{e_t}\right)e^{-\gamma z}$$
$$\mathbf{H}(t_1, t_2, z) = \left(h_z\hat{\mathbf{z}} + \mathbf{h_t}\right)e^{-\gamma z}$$
5. Do the same thing for the reflected wave, replacing γ by -γ

Helmholtz' equation

$$\left(\nabla_t^2 + k_c^2\right) \mathrm{e}_z = 0 \;;\; \left(\nabla_t^2 + k_c^2\right) h_z = 0$$

	TEM Mode	TM or E Modes	TE or H Modes	Hybrid Modes
ez	0	not 0	0	not 0
hz	0	0	not 0	not 0

Examples

- coaxial cable : TEM, TE, TM
- optical fibres : Hybrid modes
- waveguides : TE, TM

TEM mode

- characteristic equation :

$$e_z = h_z = 0$$

- second stage :

$$k_c^2 \mathbf{e_t} = -\gamma \nabla_t e_z + j\omega\mu \hat{\mathbf{z}} \times \nabla_t h_z = 0$$

$$k_c^2 \mathbf{h_t} = -\gamma \nabla_t h_z - j\omega\varepsilon \hat{\mathbf{z}} \times \nabla_t e_z = 0$$

TEM Mode

- Two solutions :

$$\mathbf{e_t} = \mathbf{h_t} = 0 \ \ \text{(trivial)}$$

$$k_c = 0 \ \ => \gamma = jk = j\omega\sqrt{\varepsilon\mu}$$

Mode TEM

- We have to solve Maxwell's equations :

$$\nabla_t \times \mathbf{e_t} = 0 \qquad -\gamma\hat{\mathbf{z}} \times \mathbf{e_t} = -j\omega\mu\mathbf{h_t}$$

$$\nabla_t \times \mathbf{h_t} = 0 \qquad -\gamma\hat{\mathbf{z}} \times \mathbf{h_t} = j\omega\varepsilon\mathbf{e_t}$$

- Laplace's equation :

$$\mathbf{e_t} = -\nabla_t V \quad with \quad \nabla_t^2 V = 0$$

TEM Mode

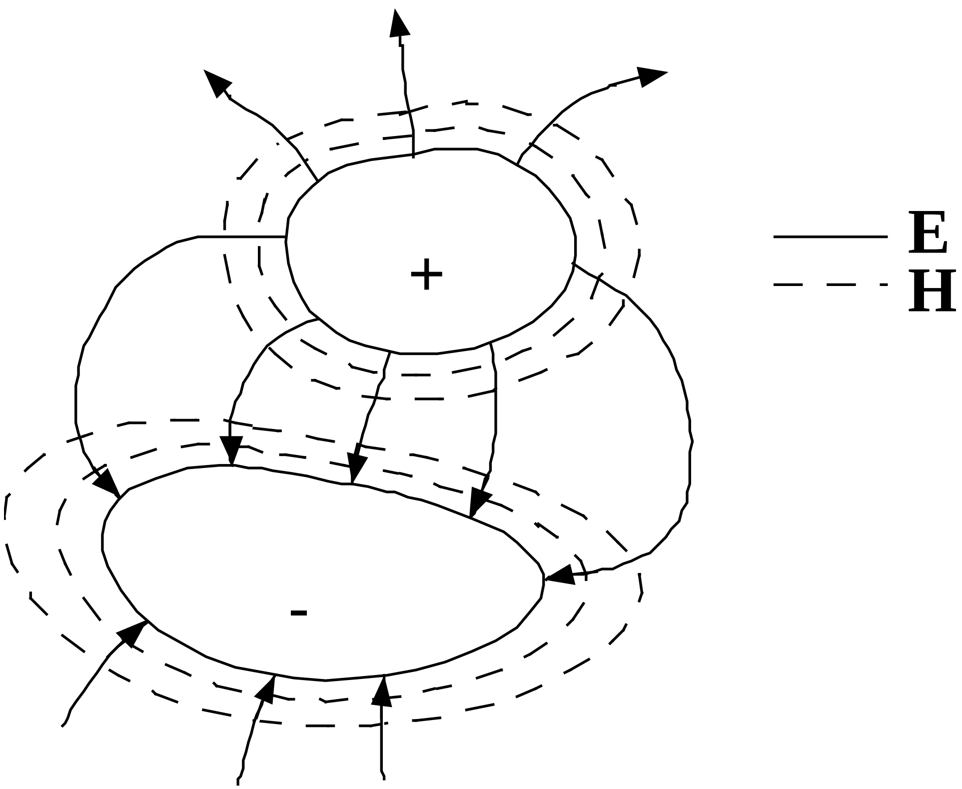
- The magnetic field is given by :

$$\mathbf{h_t} = \frac{\gamma}{j\omega\mu}\hat{\mathbf{z}} \times \mathbf{e_t} = \frac{j\omega\varepsilon}{\gamma}\hat{\mathbf{z}} \times \mathbf{e_t}$$

- The wave impedance is given by :

$$Z_{\mathrm{mod}} = \frac{j\omega\mu}{\gamma} = \frac{\gamma}{j\omega\varepsilon} = \sqrt{\frac{\mu}{\varepsilon}}$$

TEM Mode



$$V = \int_{+}^{-} \mathbf{E} \cdot \mathbf{dl}$$

$$I = \oint_{C+} \mathbf{H} \cdot \mathbf{dl}$$

$$Z_C = \frac{V}{I}$$

TEM Mode

- univoque definition of the current and voltage
- Fields identical to static fields
- zero cutoff frequency
- Needs at least two isolated conductors

TM modes

- The equation to be solved is :

$$\left(\nabla_t^2 + k_c^2\right)\mathrm{e}_z = 0$$

- The other components are obtained by :

$$k_c^2 \mathbf{e_t} = -\gamma \nabla_t e_z \quad \mathbf{h_t} = \frac{j\omega\varepsilon}{\gamma}\hat{\mathbf{z}} \times \mathbf{e_t}$$

TM modes

- Longitudinal propagation factor :

$$\gamma = \sqrt{k_c^2 - \omega^2 \varepsilon\mu}$$

- Wave impedance :

$$Z_{\mathrm{mod}} = \frac{|\mathbf{e_t}|}{|\mathbf{h_t}|} = \frac{\gamma}{j\omega\varepsilon} = \frac{\sqrt{\omega^2\mu\varepsilon - k_c^2}}{\omega\varepsilon}$$

TE modes

- The equation to be solved is :

$$\left(\nabla_t^2 + k_c^2\right) h_z = 0$$

- The other components are obtained by :

$$k_c^2 \mathbf{h_t} = -\gamma \nabla_t h_z \quad \mathbf{e_t} = \frac{j\omega\mu}{\gamma} \mathbf{h_t} \times \hat{\mathbf{z}}$$

TE modes

- Longitudinal propagation factor :

$$\gamma = \sqrt{k_c^2 - \omega^2 \varepsilon\mu}$$

- Wave impedance :

$$Z_{\mathrm{mod}} = \frac{|\mathbf{e_t}|}{|\mathbf{h_t}|} = \frac{j\omega\mu}{\gamma} = \frac{\omega\mu}{\sqrt{\omega^2\mu\varepsilon - k_c^2}}$$

Longitudinal propagation coefficient

$$\gamma = \alpha + j\beta = k_c\sqrt{1-\left(\frac{\omega}{\omega_c}\right)^2} = \frac{\omega_c}{c}\sqrt{1-\left(\frac{\omega}{\omega_c}\right)^2}$$

Group velocity

$$v_g = \left(\frac{\partial\beta}{\partial\omega}\right)^{-1} = c\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}$$

phase velocity

$$v_\varphi = \frac{\omega}{\beta} = \frac{c}{\sqrt{1-\left(\dfrac{\omega_c}{\omega}\right)^2}}$$

guided wavelength

$$\lambda_g = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{k^2 - k_c^2}} = \frac{\lambda}{\sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^2}}$$

with

$$\lambda_c = \frac{2\pi}{k_c}$$

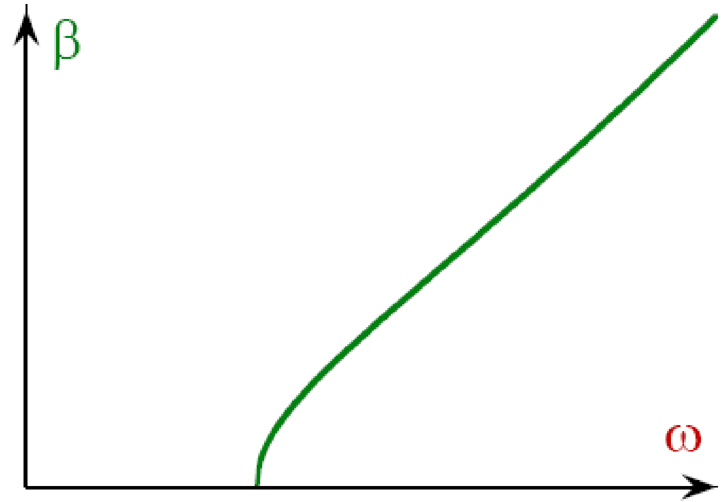
	TEM Modes	TM Modes	TE Modes
Basic equation	$\nabla_t^2 V = 0$	$\left(\nabla_t^2 + k_c^2\right) e_z = 0$	$\left(\nabla_t^2 + k_c^2\right) h_z = 0$
γ	$\sqrt{-\omega^2 \varepsilon\mu} = j\omega\sqrt{\varepsilon\mu}$	$\sqrt{k_c^2 - \omega^2 \varepsilon\mu}$	$\sqrt{k_c^2 - \omega^2 \varepsilon\mu}$
Ez	0	$e_z \mathrm{e}^{-\gamma z}$	0
Hz	0	0	$h_z \mathrm{e}^{-\gamma z}$
Et	$-\nabla_t V \mathrm{e}^{-\gamma z}$	$-\left(\dfrac{\gamma}{k_c^2}\right)\nabla_t E_z$	$\left(\dfrac{j\omega\mu}{\gamma}\right)(\mathbf{H_t} \times \hat{\mathbf{z}})$
Ht	$\left(\dfrac{\gamma}{j\omega\mu}\right)(\hat{\mathbf{z}} \times \mathbf{E_t})$	$\left(\dfrac{j\omega\varepsilon}{\gamma}\right)(\hat{\mathbf{z}} \times \mathbf{E_t})$	$-\left(\dfrac{\gamma}{k_c^2}\right)\nabla_t H_z$
Zmod = |**et**| / |**ht**|	$\sqrt{\dfrac{\mu}{\varepsilon}}$	$\dfrac{\sqrt{\omega^2 \varepsilon\mu - k_c^2}}{\omega\varepsilon}$	$\dfrac{\omega\mu}{\sqrt{\omega^2 \varepsilon\mu - k_c^2}}$

Dispersion and distorsion (1)

When the linear phase shift β is a non-linear function of the frequency, the propagation is sais to be dispersive.

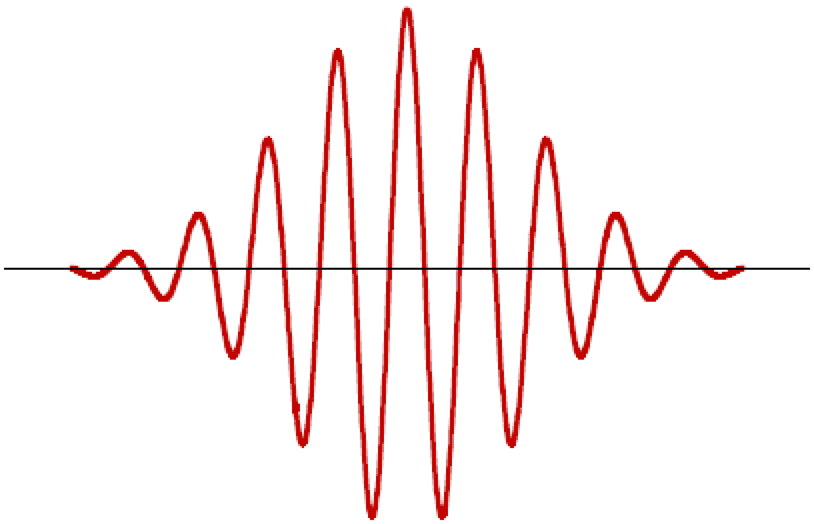
Waveguides and optical fibres are dispersive propagation lines

When a signal propagates on a dispersive line, it sustains a distorsion.



This effect is illustrated here with a "gaussian pulse" — where the developpments is mathematically simple

Dispersion and distorsion (2)



Modulated Gaussian pulse

$$f(t, z=0) = \cos(\omega_0 t)\,e^{-\frac{1}{2}\left(t/\tau\right)^2}$$

where 2τ is the width of the pulse at a level $1/\sqrt{e} = 0,606$ of the maximum.

The Fourier transform of the gaussian pulse (spectrum of the pulse in the frequency domain) has also a gaussian dependency

$$\underline{F}(\omega, z=0) = \tau\sqrt{2\pi}\,e^{-\frac{1}{2}\left[\tau(\omega-\omega_0)\right]^2}$$

Dispersion and distorsion (3)

The signal propagates along the dispersive (dispersive) line and sustains a phase shift βz

$$\underline{F}(\omega, z) = \tau\sqrt{2\pi}\, e^{-\frac{1}{2}[\tau(\omega-\omega_0)]^2} e^{-j\beta z}$$

The corresponding function in time domain is found by taking the inverse Fourier transform

$$f(t,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underline{F}(\omega,z)e^{j\omega t}d\omega = \frac{\tau\sqrt{2\pi}}{2\pi}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}[\tau(\omega-\omega_0)]^2} e^{-j\beta z}e^{j\omega t}d\omega$$

But β is not a simplefunction of ω, thus this integral cannot be evaluated in a "simple" way ...

Dispersion and distorsion (4)

The spectrum of the pulse is usually narrow, thus we take a Taylor series developpment of $\beta(\omega)$ around ω_0

$$\beta = \beta_0 + \beta_1(\omega-\omega_0) + \frac{\beta_2}{2}(\omega-\omega_0)^2 + \ldots \quad \text{avec} \quad \beta_0 = \beta(\omega_0) \text{ et } \beta_n = \left.\frac{\partial^n \beta}{\partial \omega^n}\right|_{\omega=\omega_0}$$

The term under the integral takes the form

$$e^{-\frac{1}{2}[\tau(\omega-\omega_0)]^2} e^{-j\left[\beta_0+\beta_1(\omega-\omega_0)+\frac{\beta_2}{2}(\omega-\omega_0)^2\right]z} e^{j\omega t}$$

regrouping the terms in ω, we obtain a term $e^{j\omega t}e^{-j\beta_1\omega z} = e^{j\omega t'}$, with $t' = t - \beta_1 z$, which corresponds to a translation with a velocity $1/\beta_1 = v_g$ (group velocity).

Dispersion and distorsion (5)

Grouping the terms in $(\omega - \omega_0)^2$, we get

$$e^{-\frac{1}{2}[\tau(\omega-\omega_0)]^2 - j\left[\frac{\beta_2}{2}(\omega-\omega_0)^2\right]z} = e^{-\frac{(\omega-\omega_0)^2}{2}\left[\tau^2 + j\beta_2 z\right]} = e^{-\frac{(\omega-\omega_0)^2 \tau_e^2}{2}}$$

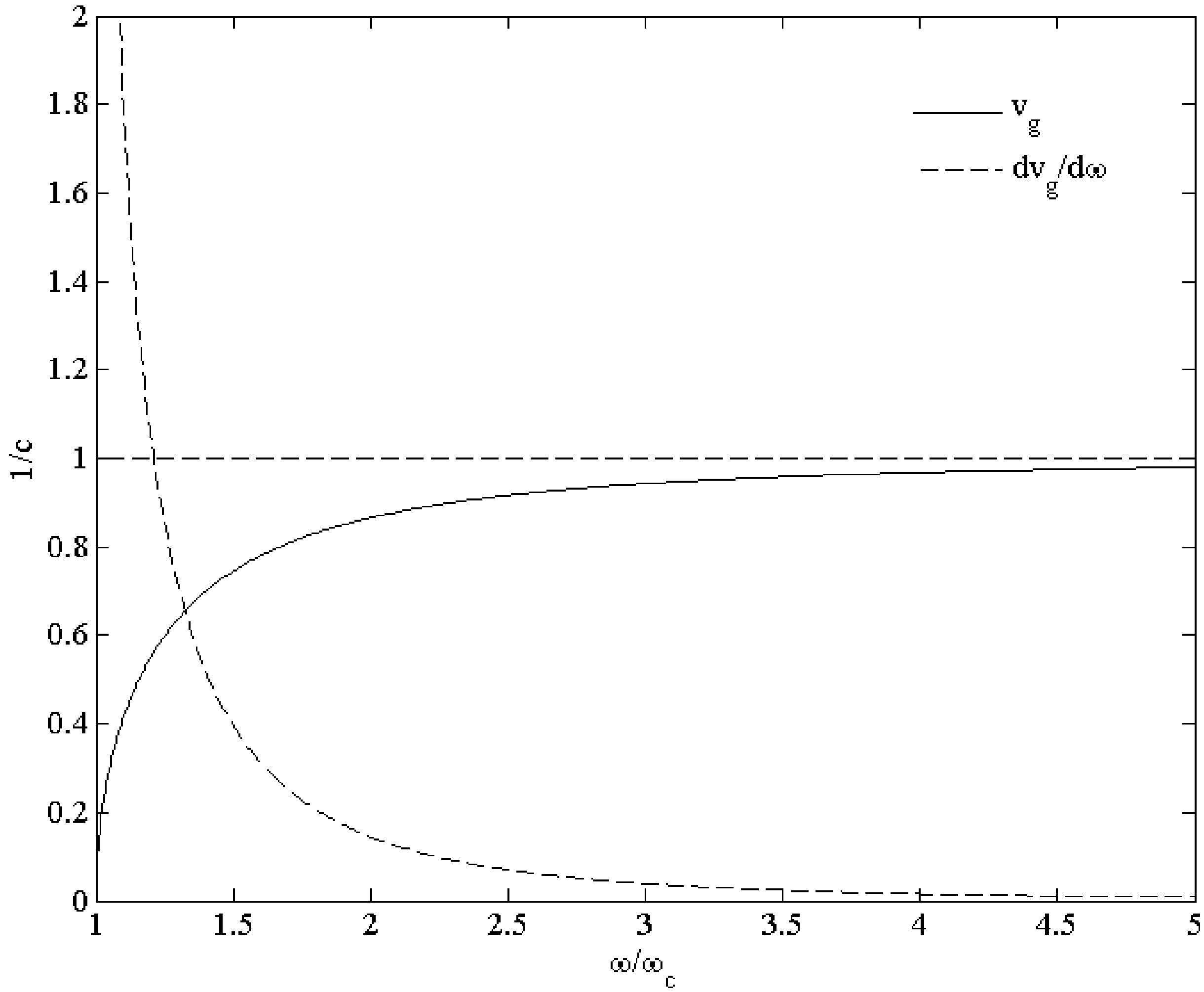
Taking the inverse Fourier transfom by making some approximations, we find that the width of the pulse becomes

$$\tau' \cong \frac{\left|\tau_e^2\right|}{\tau} = \sqrt{\tau^2 + \left(\frac{\beta_2 z}{\tau}\right)^2}$$

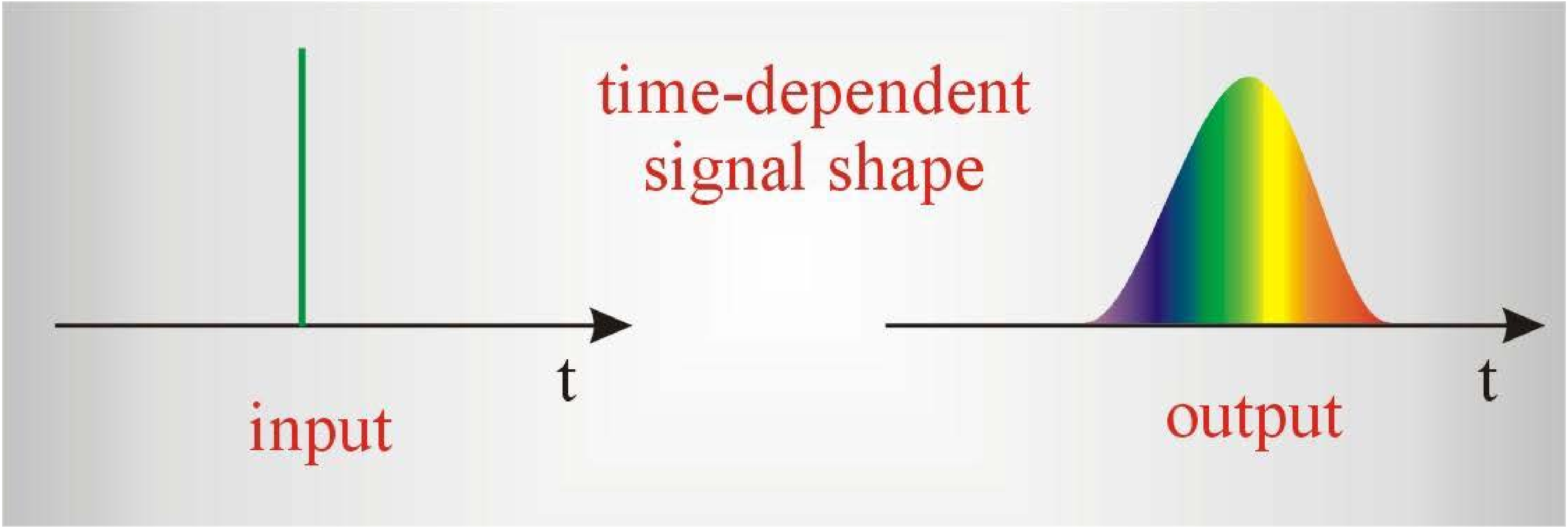
At a large distance z, if the dispersion term β_2 is important, we tend towards $\tau' \cong \beta_2 z/\tau$: a very narrow pulse in $z = 0$ broadens faster then a wider pulse in $z = 0$.

$$\beta_2 = -\frac{\mathrm{d}v_g/\mathrm{d}\omega}{v_g^2}$$

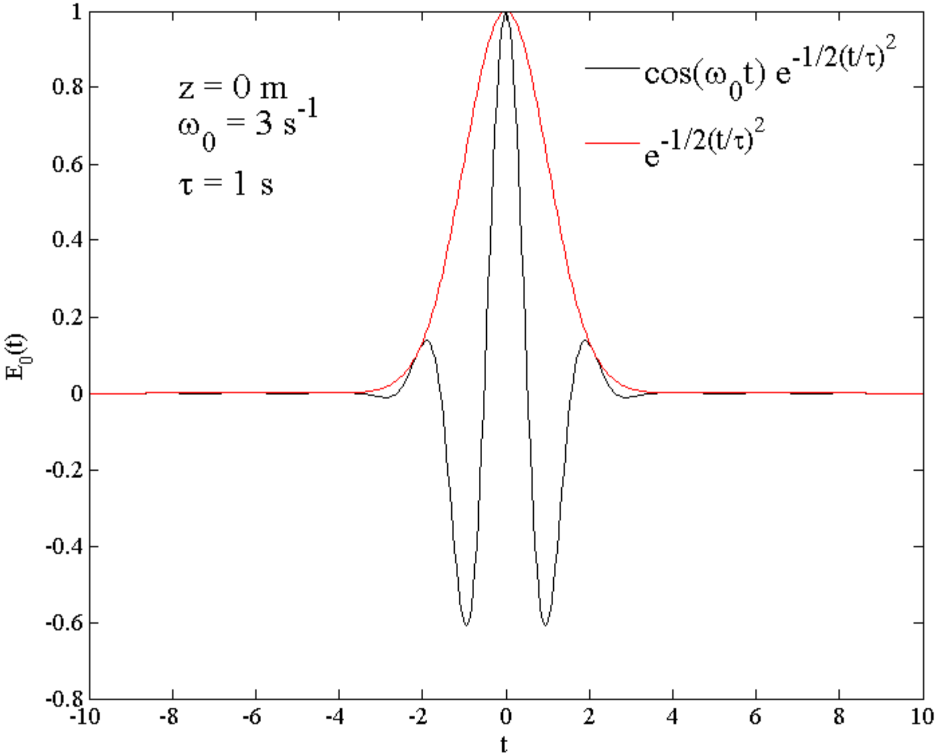
Dispersion and distorsion (6)

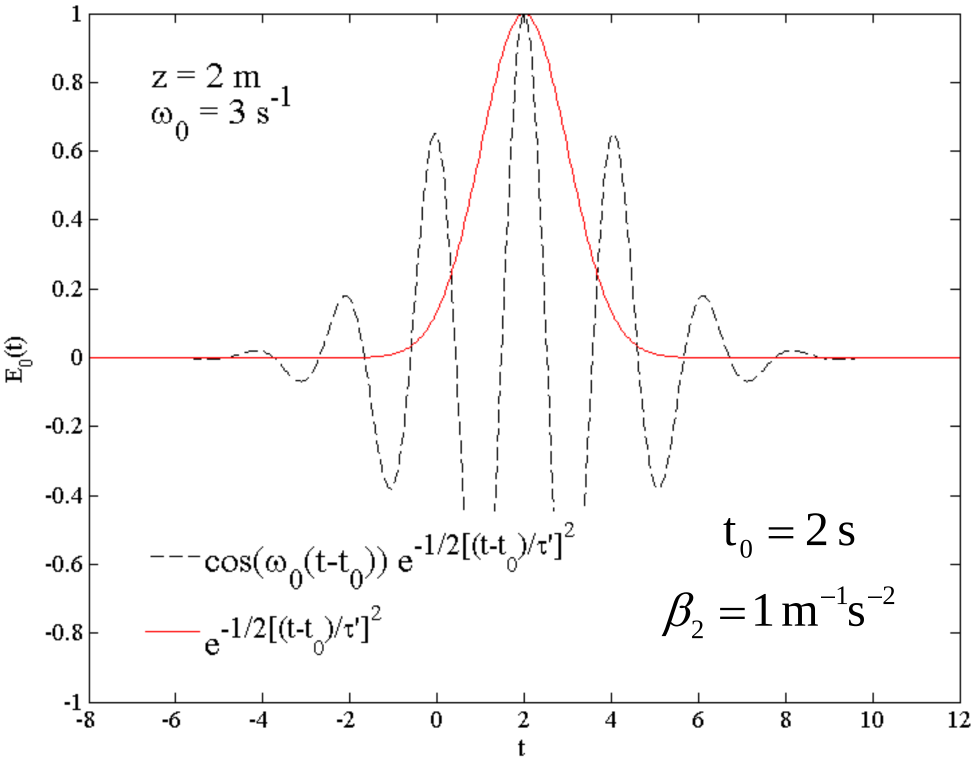


Dispersion and distorsion (7)



Dispersion and distorsion (8)



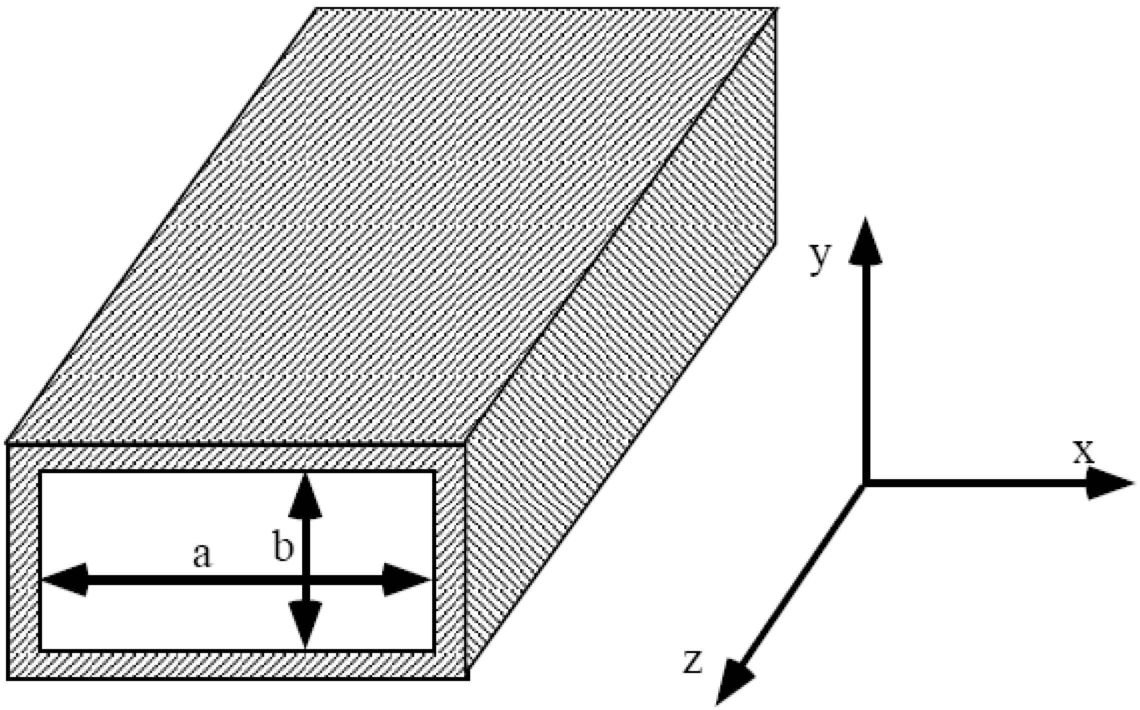


$t_0 = 2$ s

$\beta_2 = 1\ \mathrm{m^{-1} s^{-2}}$

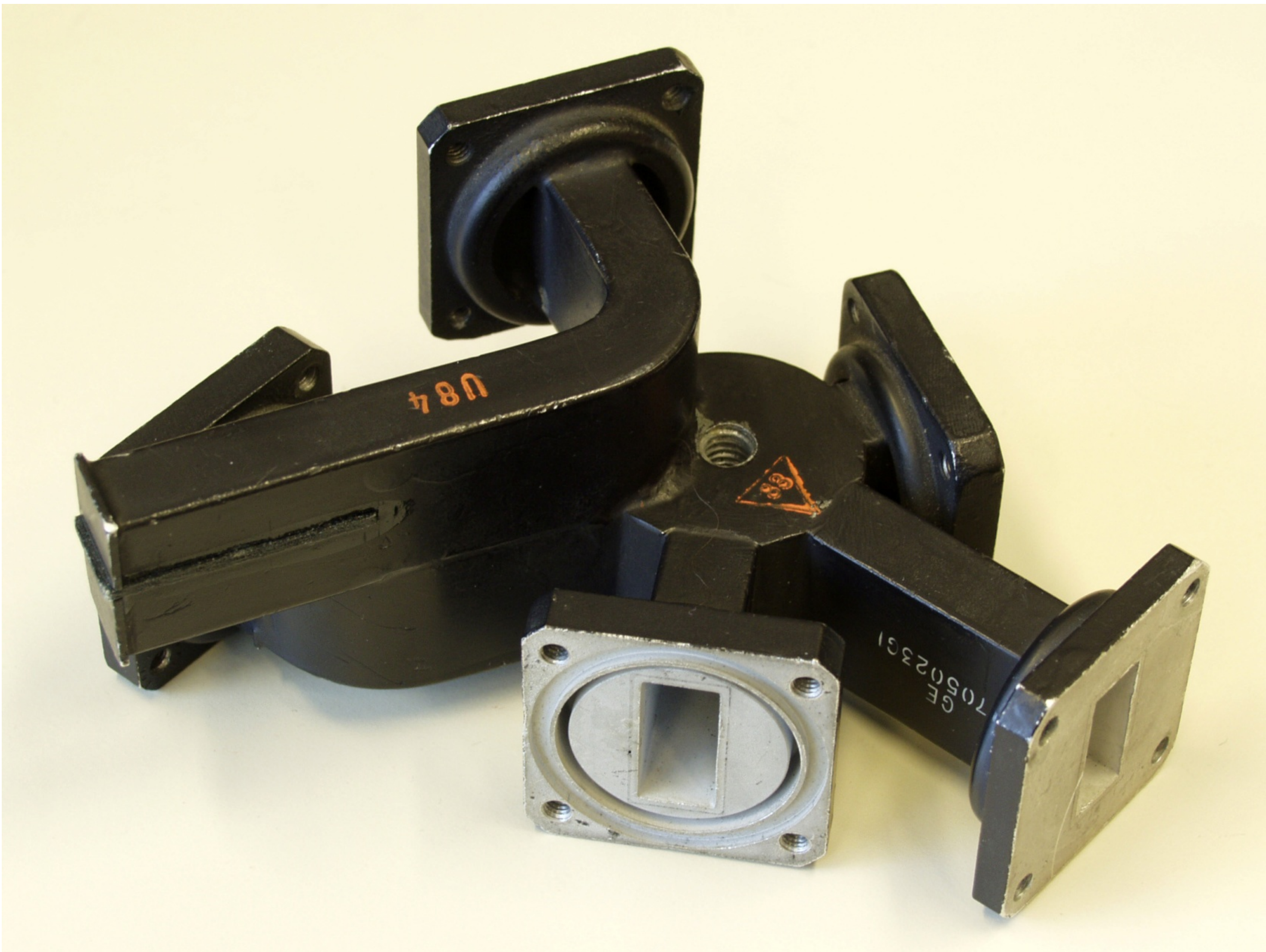
$$\tau' = \sqrt{\tau^2 + \left(\frac{\beta_2 z}{\tau}\right)^2}$$

Rectangular waveguides



Rectangular waveguides

Rectangular waveguides



Rectangular waveguides

Rectangular waveguides : TEM Modes

We need to solve Laplace's equation

$$\nabla_t^2 V = 0$$

with the following boundary conditions

$$V\big|_{x=0,x=a,y=0,y=b} = cte$$

As the guide boundaries are connex, this means that $V = cte$

and thus $\mathbf{e_t} = \nabla_t V = 0$

No TEM Mode

Rectangular waveguides : TM Modes

Basic equation $\left(\nabla_t^2 + k_c^2\right)\mathrm{e}_z = \dfrac{\partial^2 \mathrm{e}_z}{\partial x^2} + \dfrac{\partial^2 \mathrm{e}_z}{\partial y^2} + k_c^2\,\mathrm{e}_z = 0$

separation of variables $\mathrm{e}_z = \mathrm{e}_{zx}(x)\,\mathrm{e}_{zy}(y)$

Thus $$\frac{d^2\,\mathrm{e}_{zx}}{dx^2}\frac{1}{\mathrm{e}_{zx}} + \frac{d^2\,\mathrm{e}_{zy}}{dy^2}\frac{1}{\mathrm{e}_{zy}} = -k_c^2$$

$-k_x^2$ $-k_y^2$

$$k_x^2 + k_y^2 = k_c^2$$

Rectangular waveguides : TM Modes

We need to solve $\dfrac{d^2 e_{zx}}{dx^2}\dfrac{1}{e_{zx}} = -k_x^2 \;,\; \dfrac{d^2 e_{zy}}{dy^2}\dfrac{1}{e_{zy}} = -k_y^2 \;,\; k_x^2 + k_y^2 = k_c^2$

Solutions :

$$e_{zx} = A\cos k_x x + B\sin k_x x$$

$$e_{zy} = C\cos k_y y + D\sin k_y y$$

$$k_x^2 + k_y^2 = k_c^2$$

Thus :

$$e_z = \left(A\cos k_x x + B\sin k_x x\right)\left(C\cos k_y y + D\sin k_y y\right)$$

$$k_x^2 + k_y^2 = k_c^2$$

Rectangular waveguides : TM Modes

The constants A, B, C, D, k_x and k_y are obtained using the boundary conditions :

$$e_z\big|_{x=0} = 0 \quad \Rightarrow \quad \left(A\cos k_x 0 + \cancel{B\sin k_x 0}\right)\left(C\cos k_y y + D\sin k_y y\right) = 0$$

Thus A= 0

$$e_z\big|_{y=0} = 0 \quad \Rightarrow \quad \left(A\cos k_x x + B\sin k_x x\right)\left(C\cos k_y 0 + \cancel{D\sin k_y 0}\right) = 0$$

Thus C= 0

Rectangular waveguides : TM Modes

$$e_z\big|_{x=a} = 0 \quad \Rightarrow \quad E_o \sin k_x a \sin k_y y = 0$$

Thus $k_x = m\pi/a$

$$e_z\big|_{y=b} = 0 \quad \Rightarrow \quad E_o \sin k_x x \sin k_y b = 0$$

Thus $k_y = n\pi/b$

with $mn \neq 0$

Rectangular waveguides : TM Modes

Finally $$e_z = E_o \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y$$

and $$E_z = E_o \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y\, e^{-\gamma z}$$

with $$\gamma = \sqrt{k_c^2 - k^2}$$

Rectangular waveguides : TE Modes

Basic equation $\left(\nabla_t^2 + k_c^2\right)h_z = \dfrac{\partial^2 h_z}{\partial x^2} + \dfrac{\partial^2 h_z}{\partial y^2} + k_c^2 h_z = 0$

separation of variables $h_z = h_{zx}\left(x\right)h_{zy}\left(y\right)$

Thus $$\frac{d^2 h_{zx}}{dx^2}\frac{1}{h_{zx}} + \frac{d^2 h_{zy}}{dy^2}\frac{1}{h_{zy}} = -k_c^2$$

$$-k_x^2 \qquad -k_y^2$$

$$k_x^2 + k_y^2 = k_c^2$$

Rectangular waveguides : TE Modes

We need to solve $\frac{d^2 h_{zx}}{dx^2}\frac{1}{h_{zx}} = -k_x^2$, $\frac{d^2 h_{zy}}{dy^2}\frac{1}{h_{zy}} = -k_y^2$, $k_x^2 + k_y^2 = k_c^2$

Solutions :

$$h_{zx} = A\cos k_x x + B\sin k_x x$$

$$h_{zy} = C\cos k_y y + D\sin k_y y$$

$$k_x^2 + k_y^2 = k_c^2$$

Thus :

$$h_z = \left(A\cos k_x x + B\sin k_x x\right)\left(C\cos k_y y + D\sin k_y y\right)$$

$$k_x^2 + k_y^2 = k_c^2$$

Rectangular waveguides : TE Modes

Boundary conditions applied to the electric field

$$e_x = -\frac{j\omega\mu}{k_c^2}\frac{\partial h_z}{\partial y} = -\frac{j\omega\mu k_y}{k_c^2}\left(A\cos k_x x + B\sin k_x x\right)\left(-C\sin k_y y + D\cos k_y y\right)$$

$$e_y = -\frac{j\omega\mu}{k_c^2}\frac{\partial h_z}{\partial x} = -\frac{j\omega\mu k_y}{k_c^2}\left(-A\sin k_x x + B\cos k_x x\right)\left(C\cos k_y y + D\sin k_y y\right)$$

$$e_x\big|_{y=0,y=b} = 0 \qquad \text{thus} \qquad D = 0 \quad k_y = \frac{n\pi}{b}$$

$$m + n \neq 0$$

$$e_y\big|_{x=0,x=a} = 0 \qquad \text{thus} \qquad \mathrm{B} = 0 \quad k_x = \frac{m\pi}{a}$$

Rectangular waveguides : TE Modes

Finally

$$h_z = H_o \cos\frac{m\pi}{a}x \cos\frac{n\pi}{b}y$$

$$H_z = H_o \cos\frac{m\pi}{a}x \cos\frac{n\pi}{b}y\, e^{-\gamma z}$$

$$\gamma = \sqrt{k_c^2 - \omega^2 \varepsilon\mu}$$

Rectangular waveguide

	TM Modes	TE Modes
k_c	$\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\ ,\ mn\neq 0$	$\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\ ,\ m+n\neq 0$
ω_c	$\frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}$	$\frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}$
α	$k_c\sqrt{1-\left(\frac{\omega}{\omega_c}\right)^2}\ ,\ \omega<\omega_c$	$k_c\sqrt{1-\left(\frac{\omega}{\omega_c}\right)^2}\ ,\ \omega<\omega_c$
β	$k\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}\ ,\ \omega>\omega_c$	$k\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}\ ,\ \omega>\omega_c$

Rectangular waveguide

	TM Modes	TE Modes
Ez	$E_0 \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y\, e^{-\gamma z}$	0
Hz	0	$H_0 \cos\frac{m\pi}{a}x \cos\frac{n\pi}{b}y\, e^{-\gamma z}$

Rectangular waveguide

	TM Modes	TE Modes
Ex	$-\frac{\gamma m\pi}{ak_c^2}E_0\cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\,\mathrm{e}^{-\gamma z}$	$\frac{j\omega\varepsilon n\pi}{bk_c^2}H_0\cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\,\mathrm{e}^{-\gamma z}$
Ey	$-\frac{\gamma n\pi}{bk_c^2}E_0\sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y\,\mathrm{e}^{-\gamma z}$	$-\frac{j\omega\varepsilon m\pi}{ak_c^2}H_0\sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y\,\mathrm{e}^{-\gamma z}$
Hx	$\frac{j\omega\varepsilon n\pi}{bk_c^2}E_0\sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y\,\mathrm{e}^{-\gamma z}$	$\frac{\gamma m\pi}{ak_c^2}H_0\sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y\,\mathrm{e}^{-\gamma z}$
Hy	$-\frac{j\omega\varepsilon m\pi}{ak_c^2}E_0\cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\,\mathrm{e}^{-\gamma z}$	$\frac{\gamma n\pi}{bk_c^2}H_0\cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\,\mathrm{e}^{-\gamma z}$
Z	$\sqrt{\frac{\mu}{\varepsilon}}\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}$	$\frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}}$

Rectangular waveguides

Note both TE and TM modes can only propagate for

$$\omega > \omega_c = \frac{1}{\sqrt{\varepsilon\mu}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

Dominant mode

The cutoff frequency is given for both modes by

$$\omega_c = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \quad , \qquad f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

but with

TM Modes	TE Modes
$mn \neq 0$	$m + n \neq 0$

Dominant mode

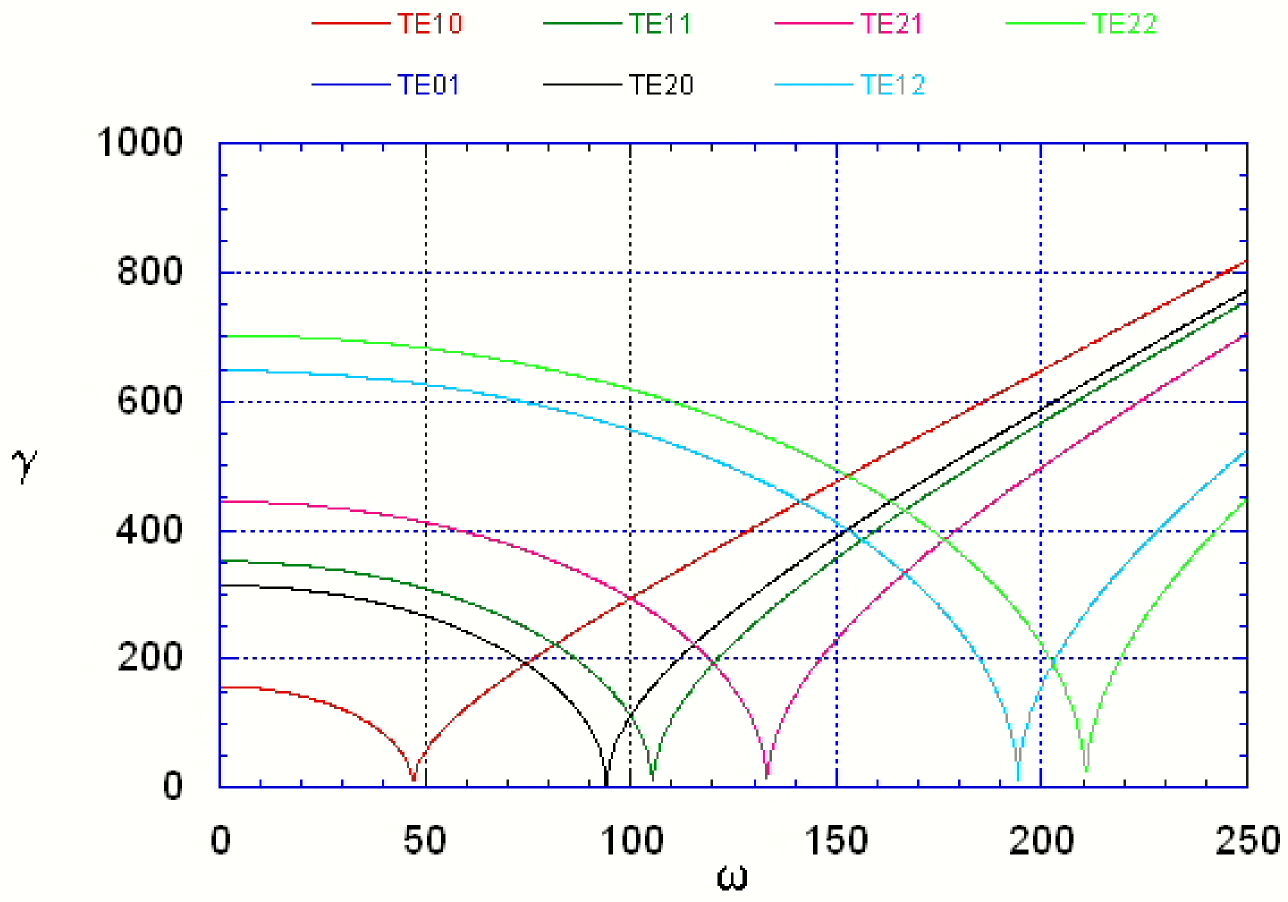
Thus the lowest possible cutoff frequency corresponds to the TE_{10} mode :

$$f_c\big|_{TE_{10}} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\frac{\pi}{a} = \frac{c}{2a}$$

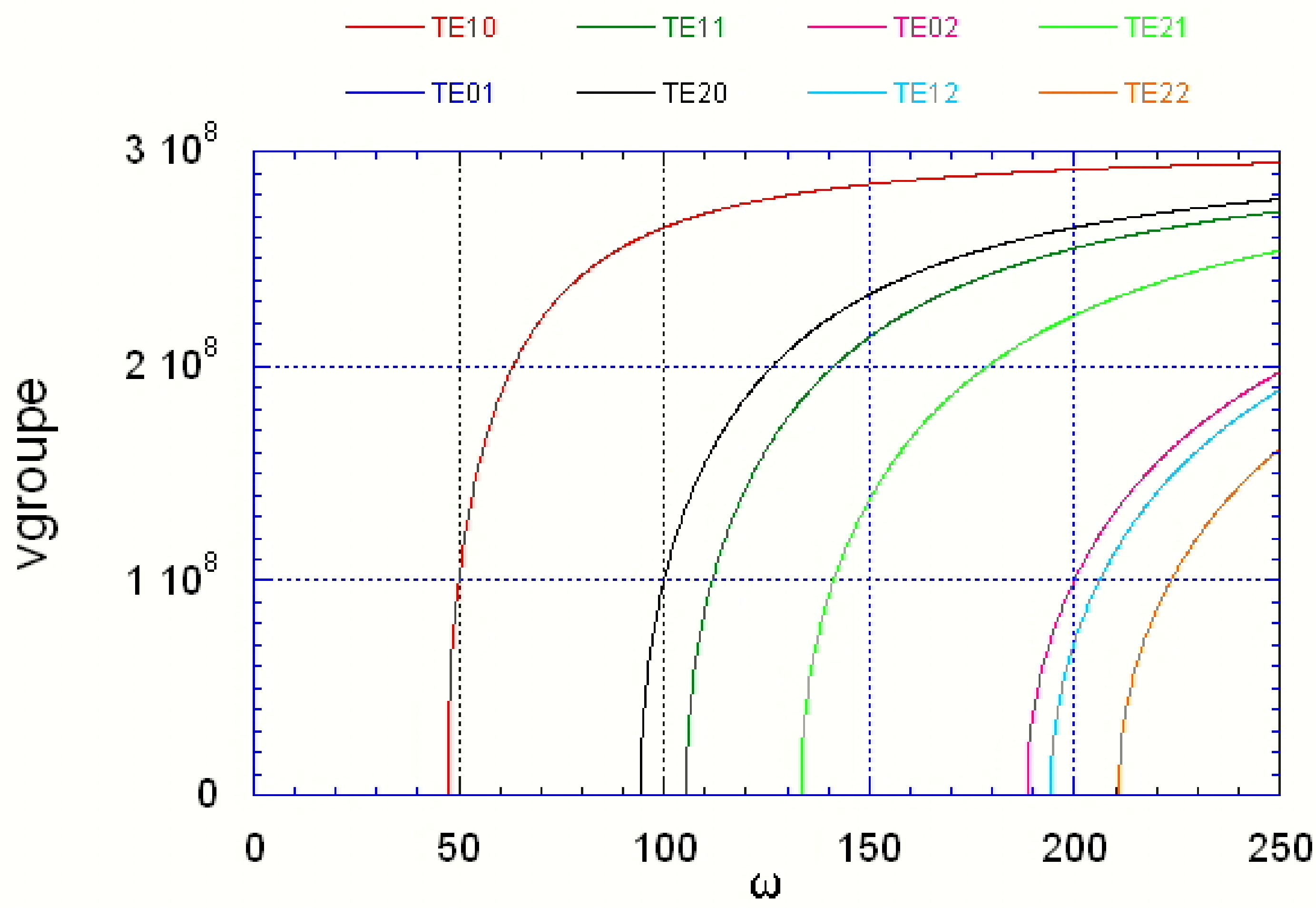
The first higher order mode is given by

$$f_c\big|_{TE_{20}} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\frac{\pi}{a} = \frac{c}{a} \qquad \text{or} \qquad f_c\big|_{TE_{01}} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\frac{\pi}{b} = \frac{c}{2b}$$

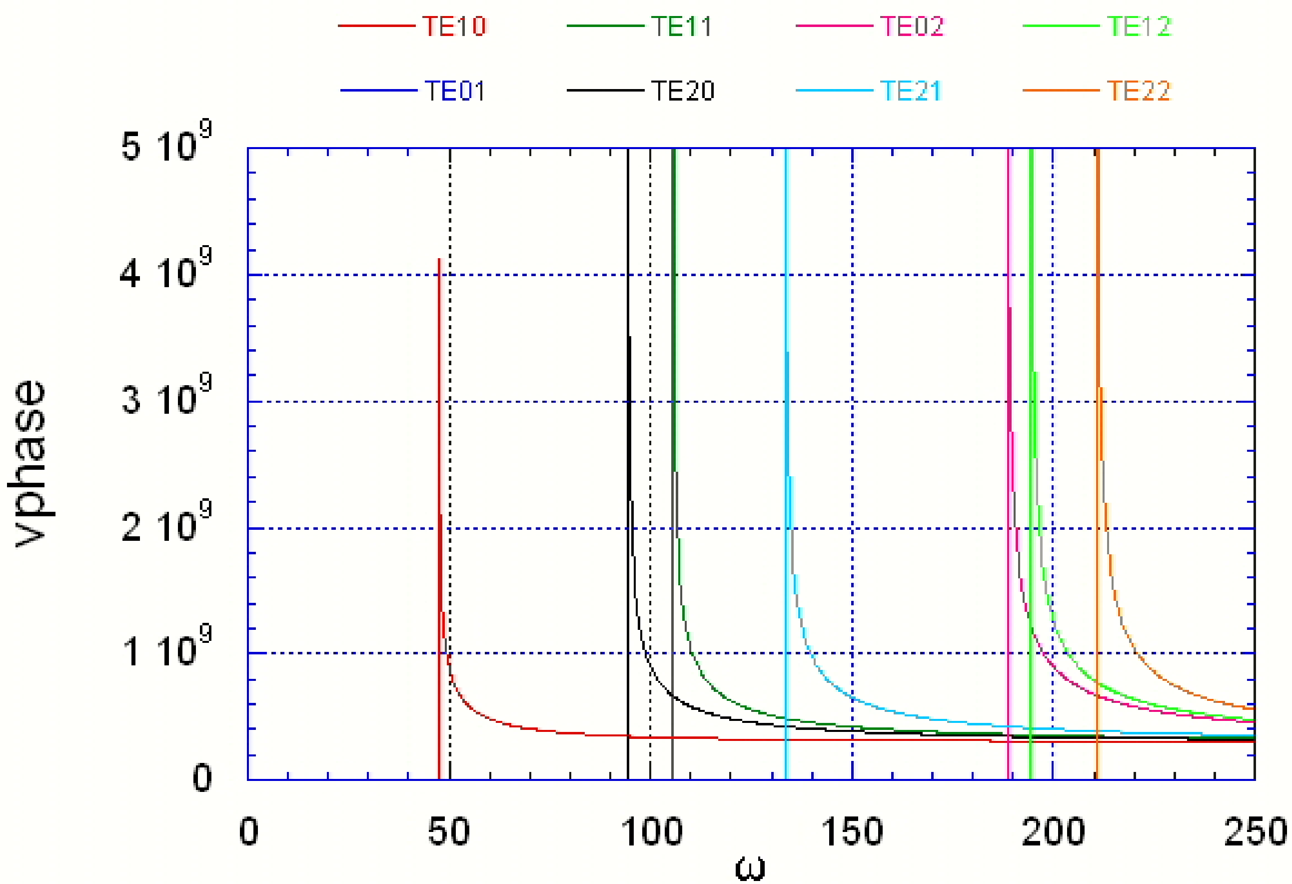
dispersion diagram



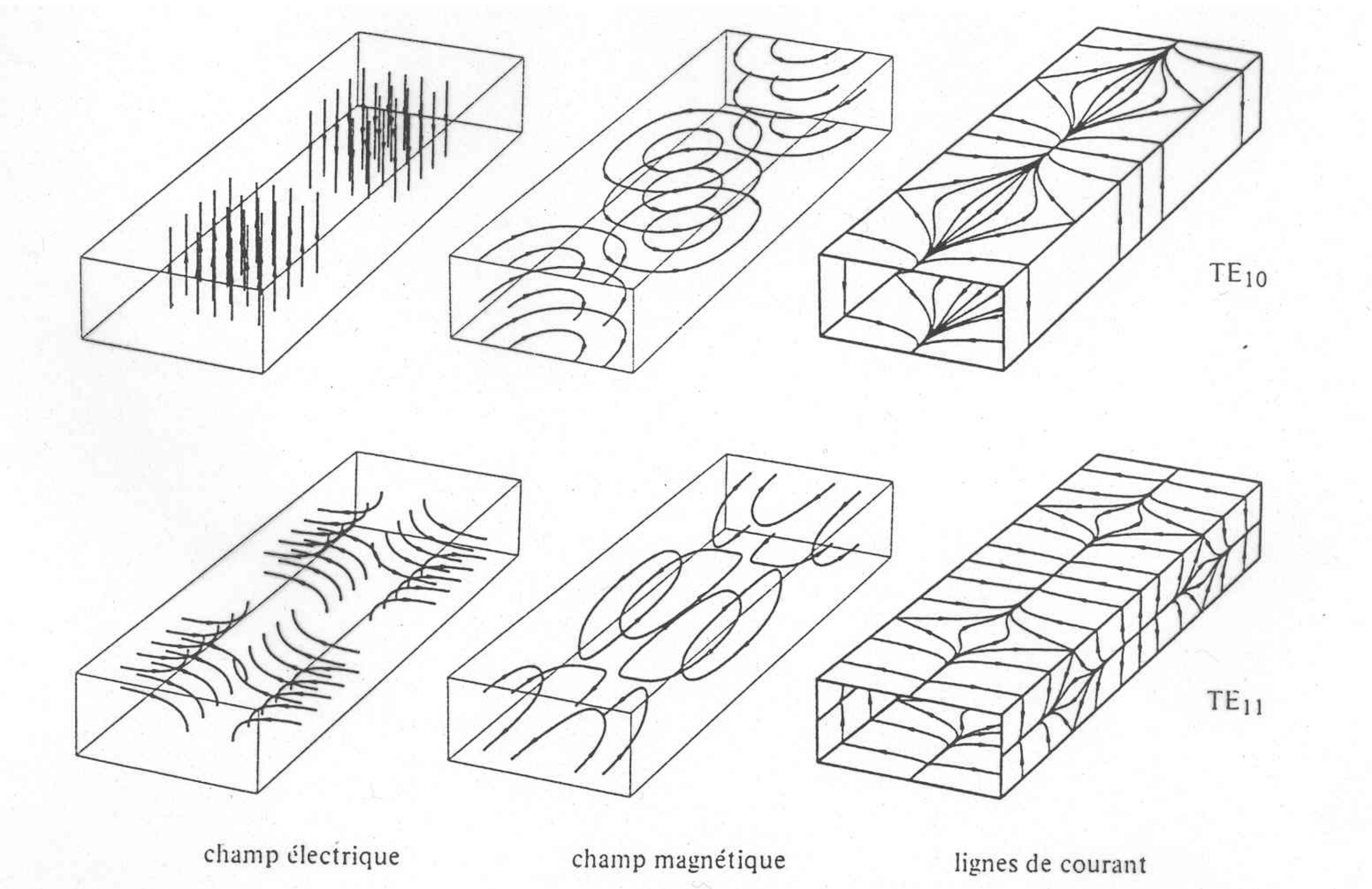
Group velocity



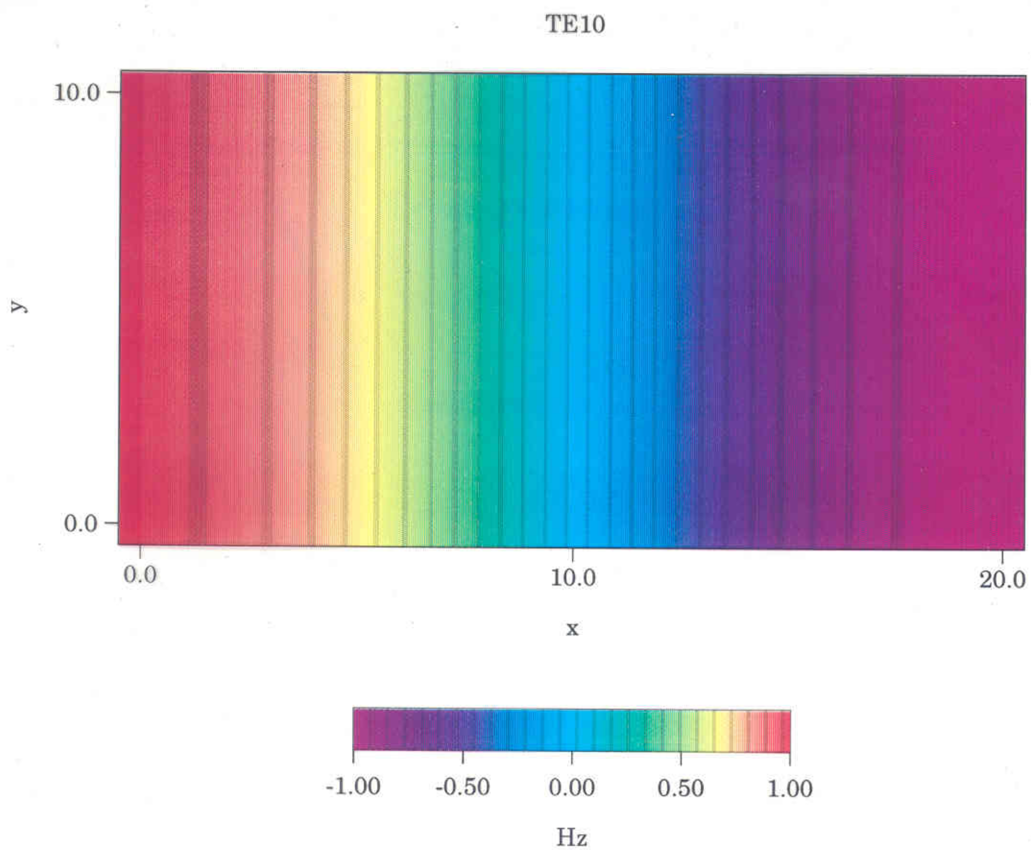
Phase velocity



Field distribution

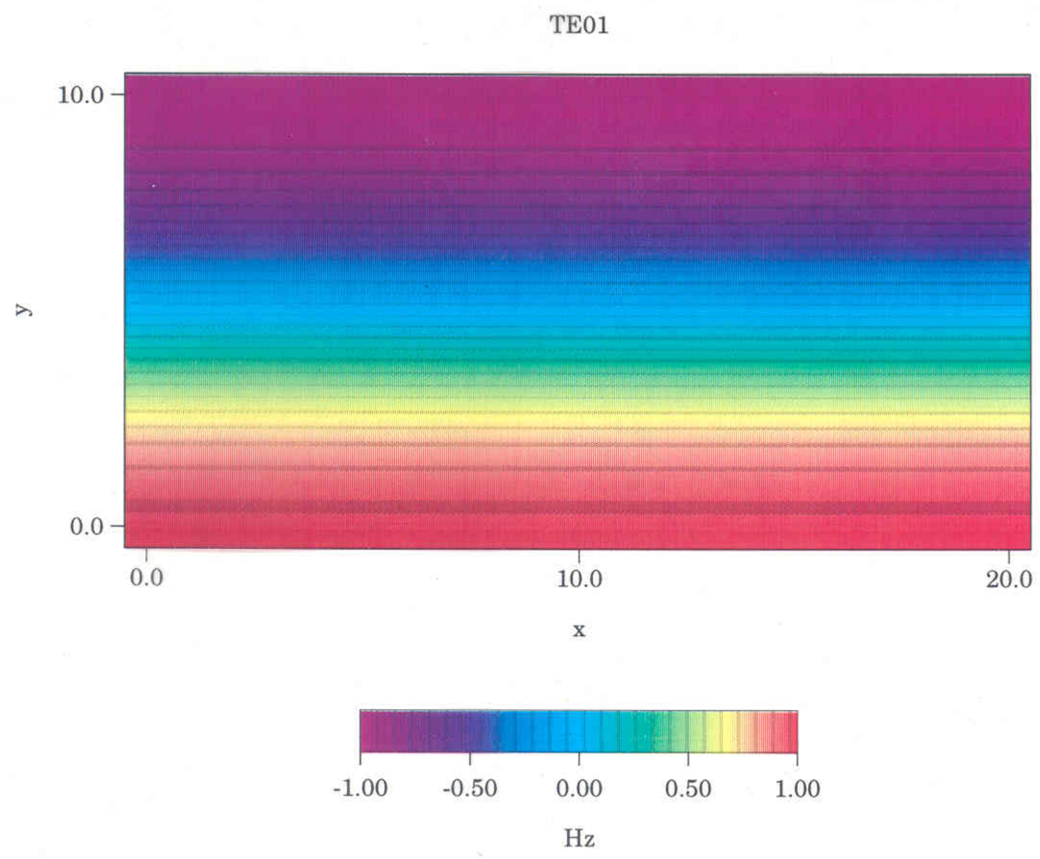


Field distribution



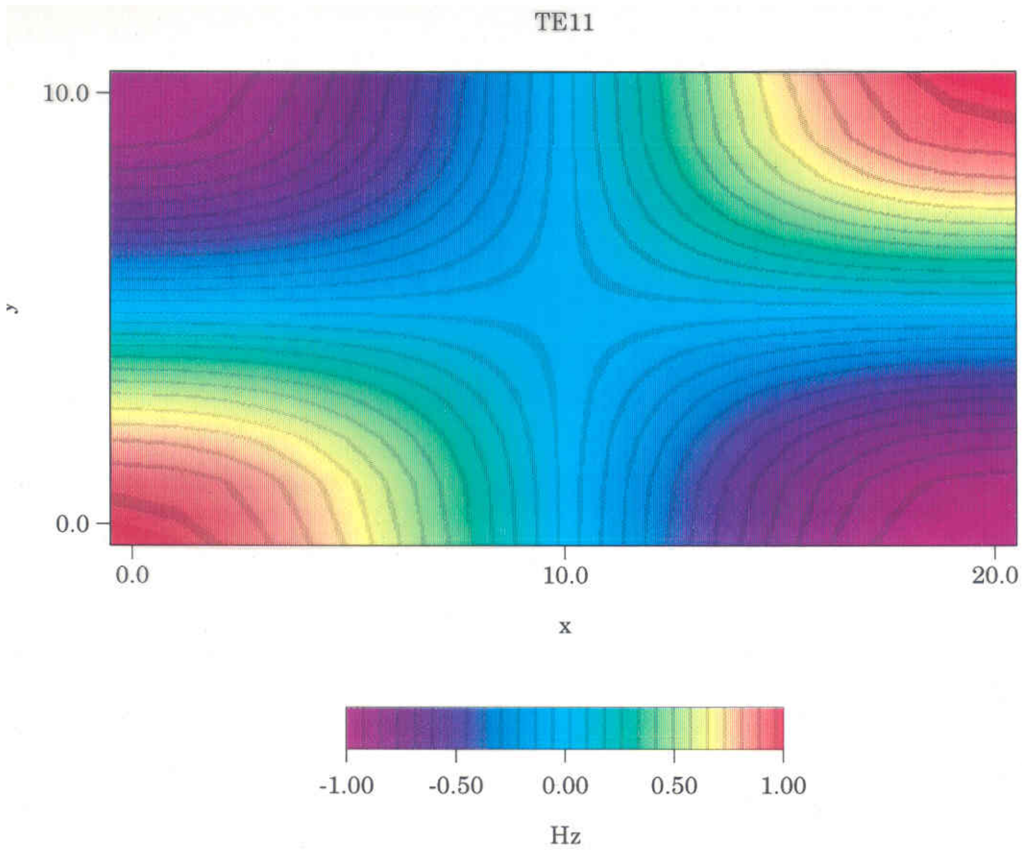
TE_{10} Hz

Field distribution



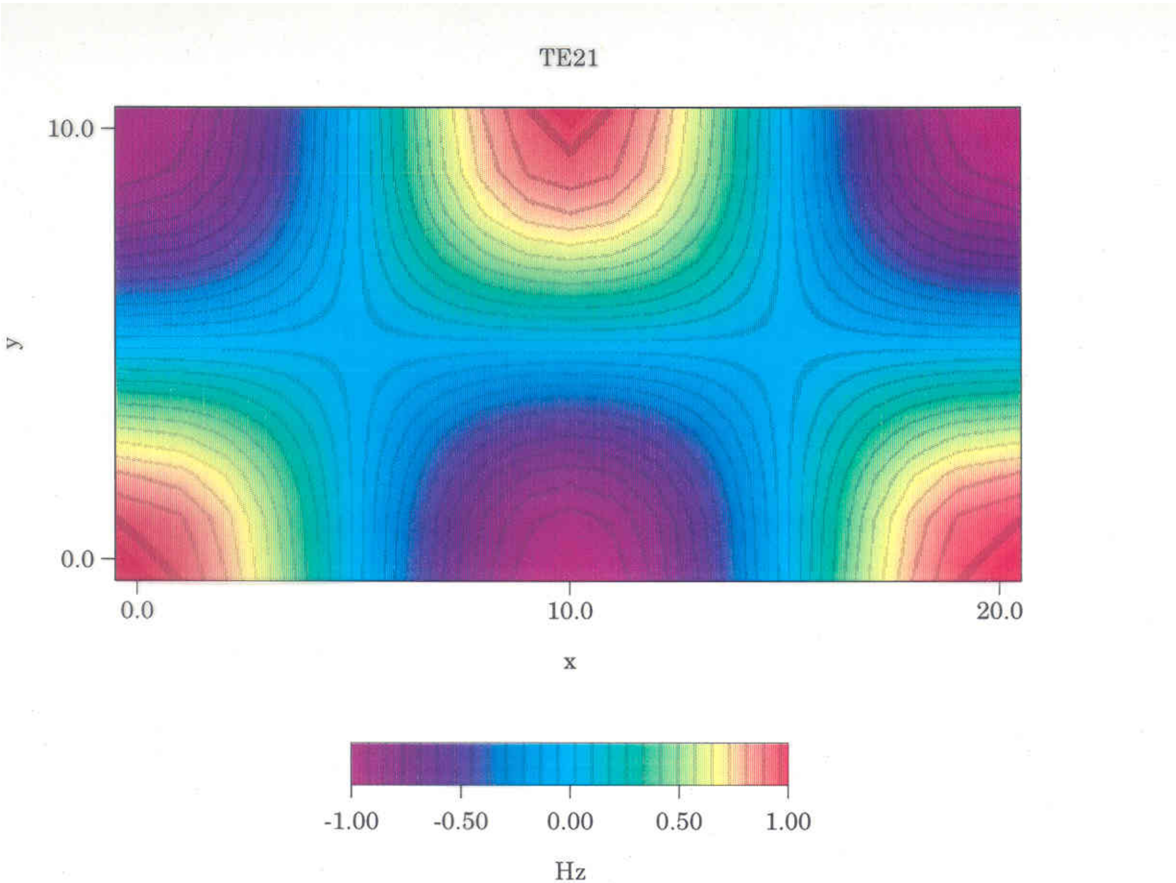
TE_{01} Hz

Field distribution



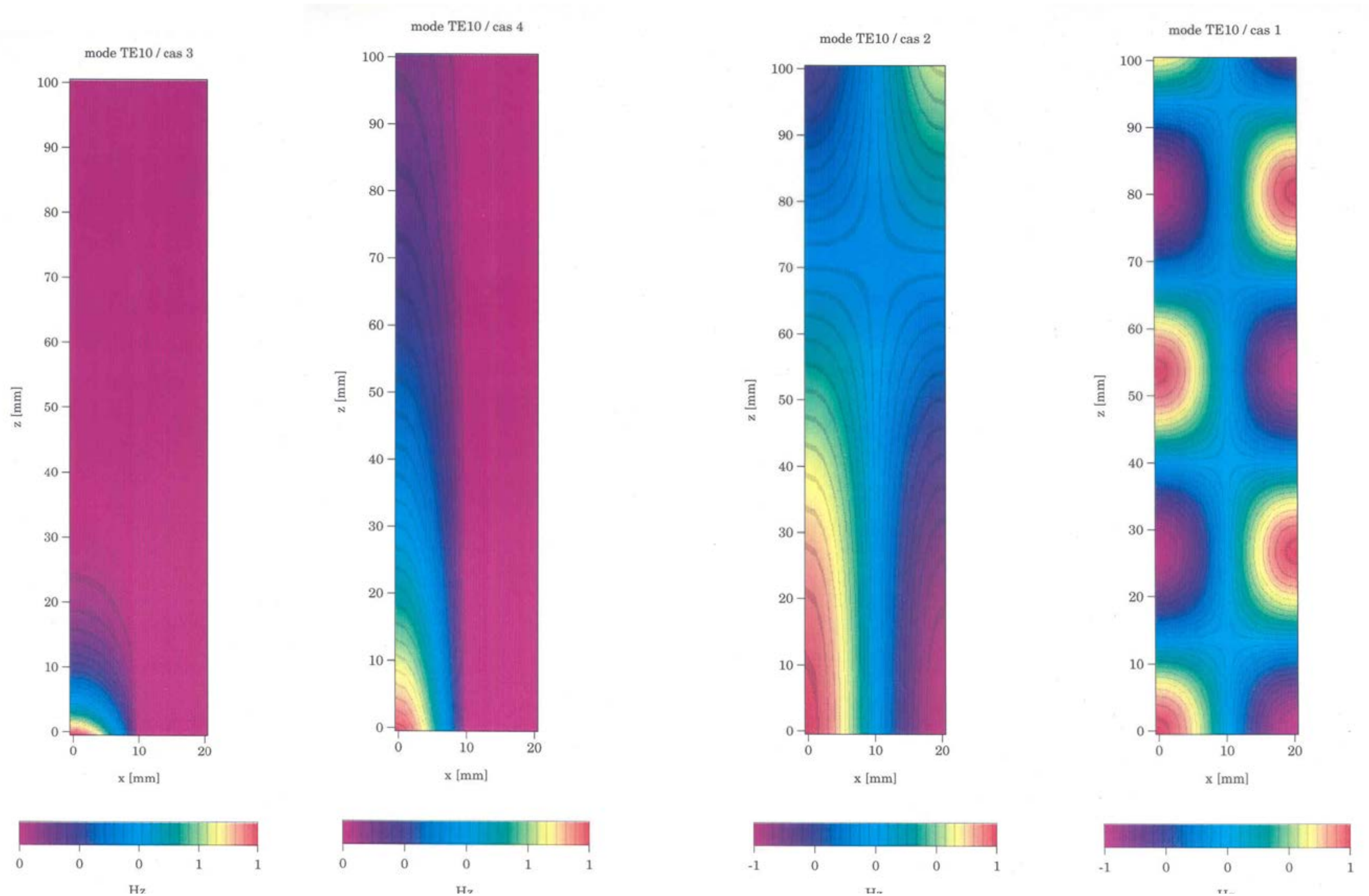
TE_{11} Hz

Field distribution



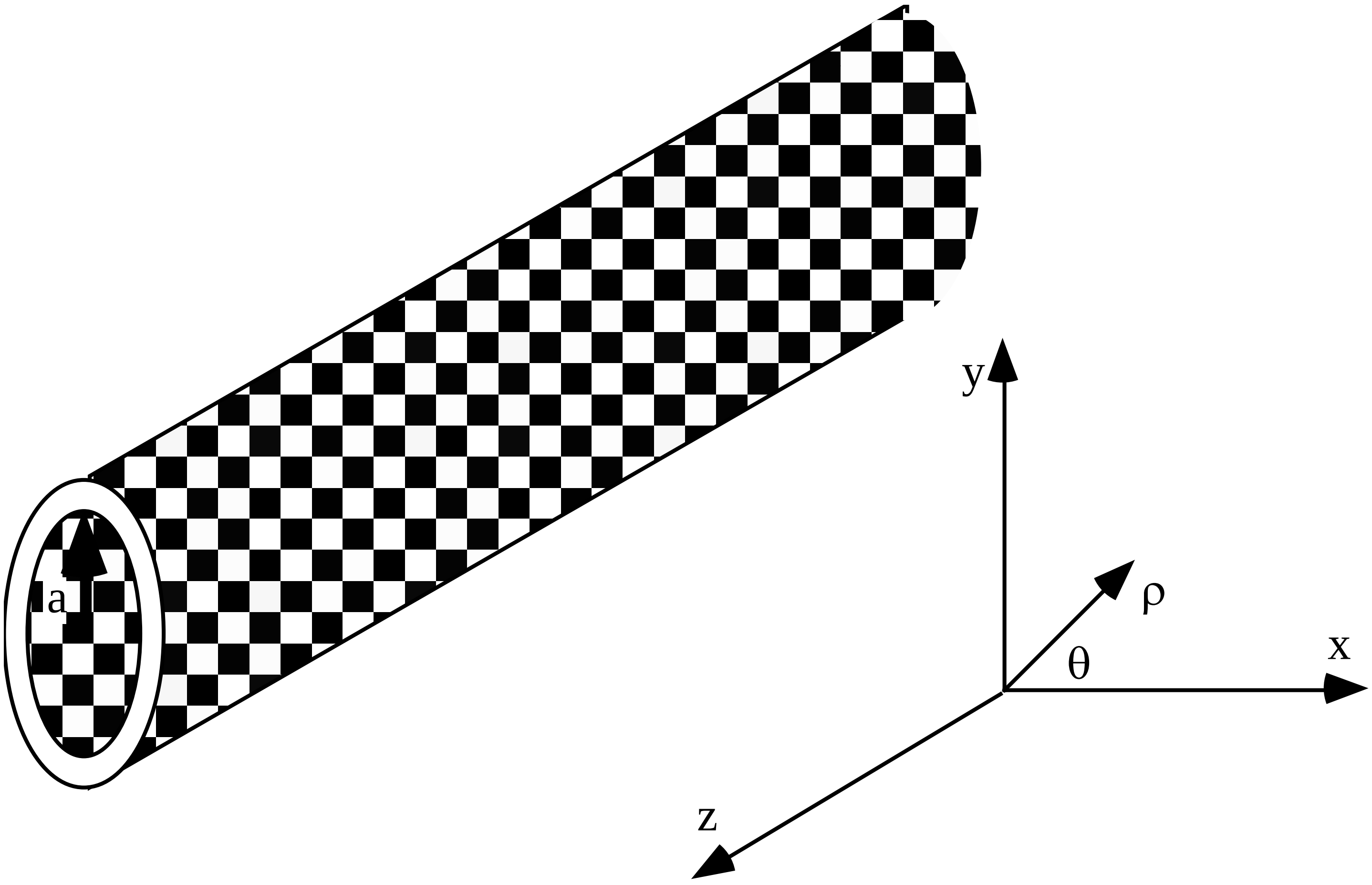
TE_{21} Hz

Field distribution



$0.95f_c$ $0.99f_c$ $1.01f_c$ $1.5f_c$

Circular waveguide



Circular waveguide

Due to the circular symmetry of the problem, cylindrical coordinates will be used :

$$e_\rho = \frac{-1}{k_c^2}\left[\gamma\frac{\partial e_z}{\partial\rho} + \frac{j\omega\mu}{\rho}\frac{\partial h_z}{\partial\varphi}\right]$$

$$e_\varphi = \frac{1}{k_c^2}\left[-\frac{\lambda}{\rho}\frac{\partial e_z}{\partial\varphi} + j\omega\mu\frac{\partial h_z}{\partial\rho}\right] \qquad k_c^2 = \gamma^2 + k_0^2 = k_0^2 - \beta^2$$

$$h_\rho = \frac{1}{k_c^2}\left[\frac{j\omega\varepsilon}{\rho}\frac{\partial e_z}{\partial\varphi} + -\gamma\frac{\partial h_z}{\partial\rho}\right]$$

$$h_\varphi = \frac{-1}{k_c^2}\left[j\omega\varepsilon\frac{\partial e_z}{\partial\rho} + \frac{\gamma}{\rho}\frac{\partial h_z}{\partial\varphi}\right]$$

TM Modes

- Wave equation in cylindrical coordinates

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+k_c^2\right)\mathrm{e}_z=0$$

- Separation of variables

$$\mathrm{e}_z(\rho,\varphi)=R(\rho)P(\varphi)$$

TM Modes

Thus
$$\frac{\rho^2}{R}\frac{d^2R}{d\rho^2}+\frac{\rho}{R}\frac{dR}{d\rho}+\rho^2k_c^2=-\frac{1}{P}\frac{d^2P}{d\varphi^2}$$

which becomes

$$-\frac{1}{P}\frac{d^2P}{d\varphi^2}=k_\varphi^2\ ,\ \frac{d^2P}{d\varphi^2}+Pk_\varphi^2=0$$

$$\frac{\rho^2}{R}\frac{d^2R}{d\rho^2}+\frac{\rho}{R}\frac{dR}{d\rho}+\rho^2k_c^2=k_\varphi^2\ \ ,\ \ \rho^2\frac{d^2R}{d\rho^2}+\rho\frac{dR}{d\rho}+R\left(\rho^2k_c^2-k_\varphi^2\right)=0$$

TM Modes

Solution of the equation in φ :

$$P(\varphi) = A\sin k_\varphi \varphi + B\cos k_\varphi \varphi$$

The solution has to be periodic in φ, thus k_φ is an integer

$$P(\varphi) = A\sin n\varphi + B\cos n\varphi$$

TM Modes

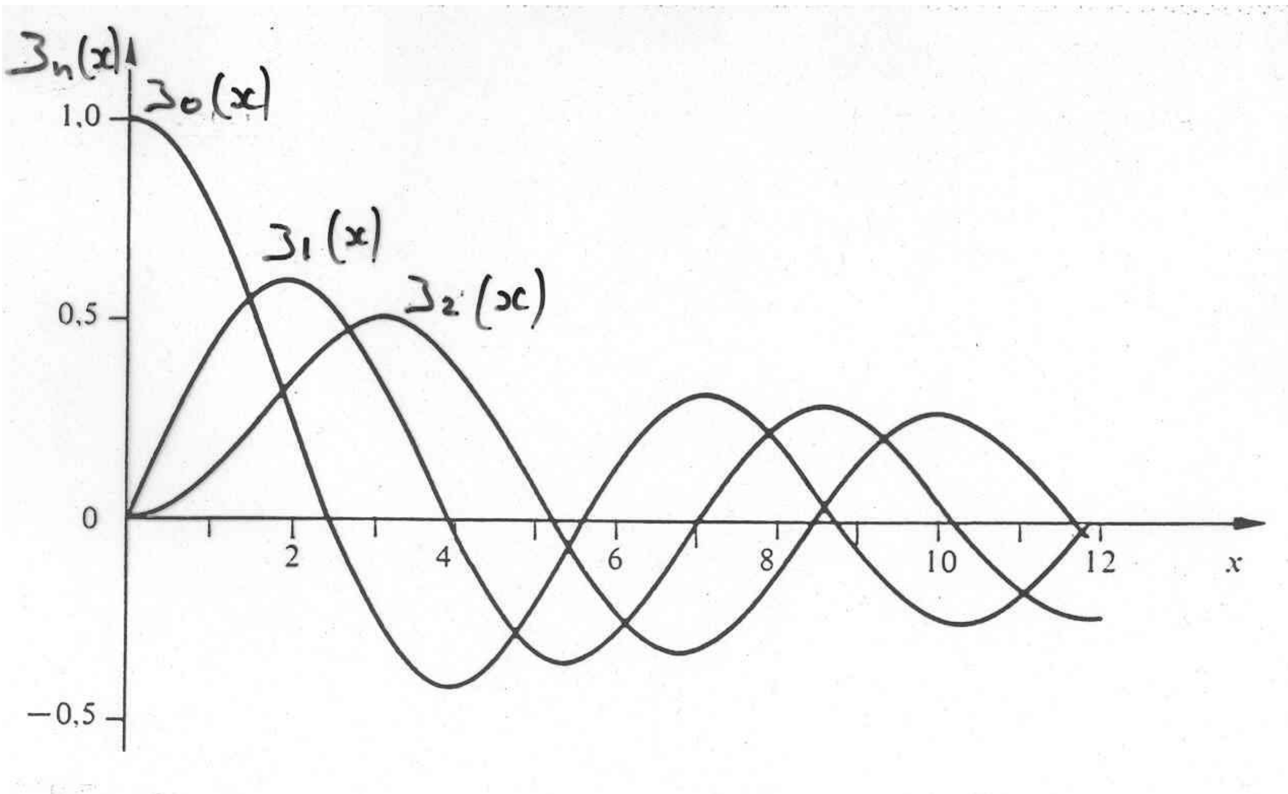
The equation in ρ takes the form of a Besel equation :

$$\rho^2 \frac{d^2 R}{d\rho^2} + \rho \frac{dR}{d\rho} + R\left(\rho^2 k_c^2 - n^2\right) = 0$$

where the solutions are given by

$$R(\rho) = CJ_n(k_c\rho) + DY_n(k_c\rho)$$

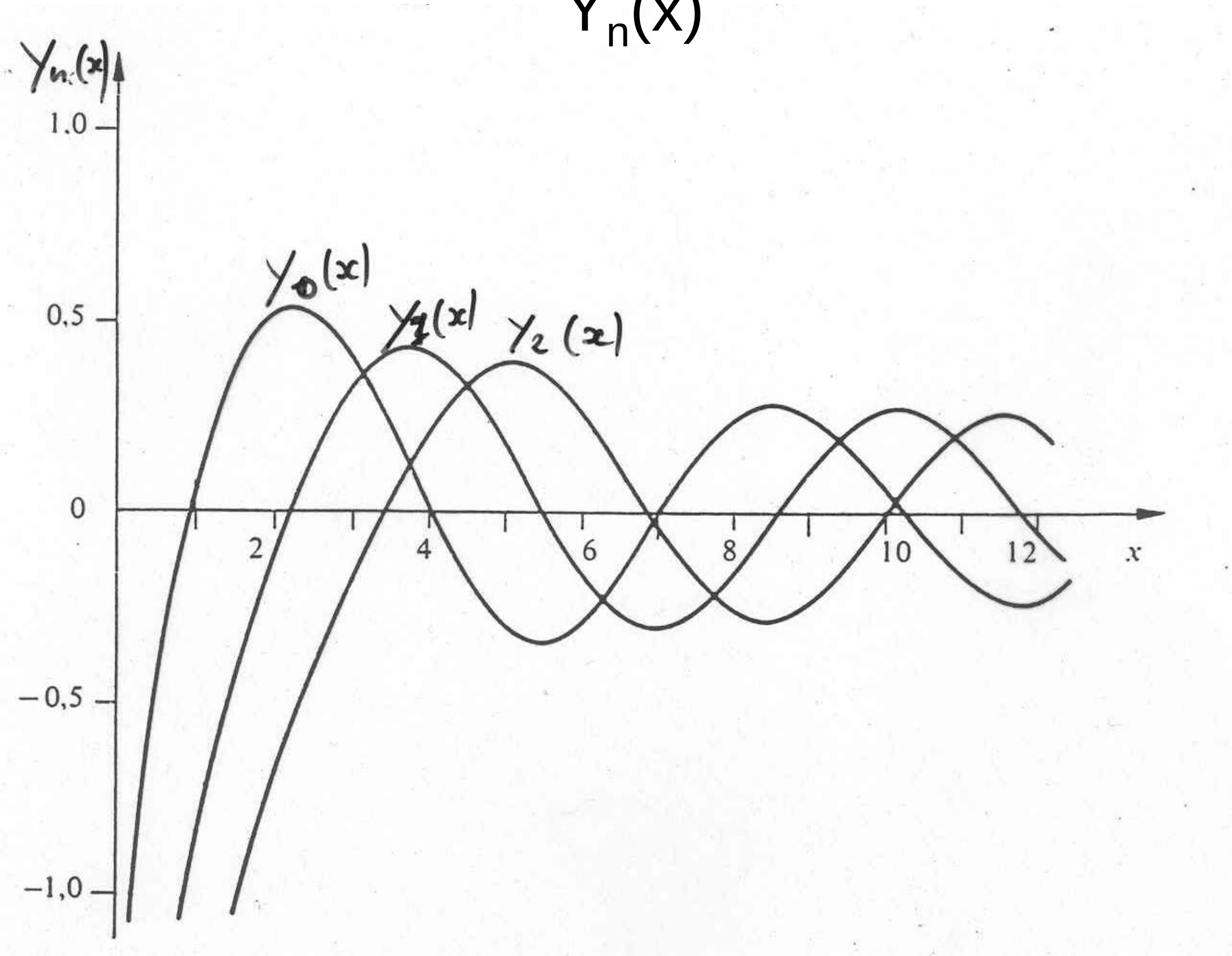
Bessel functions of the first kind



$J_n(x)$

Bessel functions of the second kind

$Y_n(x)$



D=0

TM Modes

Thus

$$e_z(\rho,\varphi) = J_n(k_c\rho)(A\sin n\varphi + B\cos n\varphi)$$

Boundary condition at ρ=a :

$$e_z\big|_{\rho=a} = J_n(k_c a)(A\sin n\varphi + B\cos n\varphi) = 0$$

Thus $k_{cnm} = \dfrac{p_{nm}}{a}$ where p_{nm} is the m^{th} zero of the Bessel function J_n

TM Modes

Propagation coeffcient

$$\beta_{nm} = \sqrt{k_0^2 - k_c^2} = \sqrt{k_0^2 - \left(\frac{p_{nm}}{a}\right)^2}$$

Cutoff frequency

$$f_{c_{nm}} = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}} = \frac{p_{nm}}{2\pi a\sqrt{\mu\varepsilon}}$$

TE Modes

Doing the same analysis than for TM modes, we obtain

$$h_z(\rho,\varphi) = J_n(k_c\rho)(A\sin n\varphi + B\cos n\varphi)$$

The boundary conditions are applied on the electric field :

$$e_\varphi(\rho,\varphi) = \frac{j\omega\mu}{k_c}(A\sin n\varphi + B\cos n\varphi) J_n'(k_c\rho)$$

where $$e_\varphi\Big|_{\rho=a} = \frac{j\omega\mu}{k_c} J'_n(k_c a)(A\sin n\varphi + B\cos n\varphi) = 0$$

TE Modes

Thus $k_{cnm} = \frac{p'_{nm}}{a}$ where p'_{nm} is the m^{th} zero of the derivative of the Bessel function of first kind J'_n

The propagation coefficient is given by

$$\beta_{nm} = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{p'_{nm}}{a}\right)^2}$$

And the cutoff frequency by $f_{c_{nm}} = \frac{p'_{nm}}{2\pi a\sqrt{\mu\varepsilon}}$

Circular waveguides

	Modes TM	Modes TE
k_c	$\frac{p_{nm}}{a}$	$\frac{p'_{nm}}{a}$
ω_c	$\frac{p_{nm}}{a\sqrt{\mu\varepsilon}}$	$\frac{p'_{nm}}{a\sqrt{\mu\varepsilon}}$
α	$k_c\sqrt{1-\left(\frac{\omega}{\omega_c}\right)^2}$, $\omega < \omega_c$	$k_c\sqrt{1-\left(\frac{\omega}{\omega_c}\right)^2}$, $\omega < \omega_c$
β	$k\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}$, $\omega > \omega_c$	$k\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}$, $\omega > \omega_c$

Circular waveguides

Ez	$J_n(k_c\rho)(A\sin n\varphi + B\cos n\varphi)e^{-\gamma z}$	0
Hz	0	$J_n(k_c\rho)(A\sin n\varphi + B\cos n\varphi)e^{-\gamma z}$

Circular waveguides

Eρ	$\frac{-\gamma}{k_c}(A\sin n\varphi + B\cos n\varphi)J_n'(k_c\rho)e^{-\gamma z}$	$-\frac{j\omega\mu n}{k_c^2\rho}(A\cos n\varphi - B\sin n\varphi)J_n(k_c\rho)e^{-\gamma z}$
Eφ	$-\frac{\gamma n}{k_c^2\rho}(A\cos n\varphi - B\sin n\varphi)J_n(k_c\rho)e^{-\gamma z}$	$\frac{j\omega\mu}{k_c}(A\sin n\varphi + B\cos n\varphi)J_n'(k_c\rho)e^{-\gamma z}$
Hρ	$\frac{j\omega\varepsilon n}{k_c^2\rho}(A\cos n\varphi - B\sin n\varphi)J_n(k_c\rho)e^{-\gamma z}$	$\frac{-\gamma}{k_c}(A\sin n\varphi + B\cos n\varphi)J_n'(k_c\rho)e^{-\gamma z}$
Hφ	$\frac{j\omega\varepsilon}{k_c}(A\sin n\varphi + B\cos n\varphi)J_n'(k_c\rho)e^{-\gamma z}$	$-\frac{\gamma n}{k_c^2\rho}(A\cos n\varphi - B\sin n\varphi)J_n(k_c\rho)e^{-\gamma z}$
Z	$\sqrt{\frac{\mu}{\varepsilon}}\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}$	$\frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}}$

Dominant mode

TE Modes $f_{c_{nm}} = \dfrac{p'_{nm}}{2\pi a\sqrt{\mu\varepsilon}}$

TM Modes $f_{c_{nm}} = \dfrac{p_{nm}}{2\pi a\sqrt{\mu\varepsilon}}$

n	p'n1	p'n2	p'n3
0	3.832	7.016	10.174
1	1.841	5.331	8.536
2	3.054	6.706	9.970
3	4.2012	6.0152	11.3459
4	5.3175	9.2824	12.6819
5	6.4156	10.5199	13.9872
6	7.5013	11.7349	15.2682

n	pn1	pn2	pn3
0	2.405	5.520	8.654
1	3.832	7.016	10.174
2	5.135	8.417	11.620
3	6.38016	9.76102	13.01520
4	7.58834	11.06471	14.37254
5	8.77142	12.33860	15.70017
6	9.93611	13.58929	17.0038

Dominant mode

TE$_{11}$ mode

$$f_{c_{nm}} = \frac{1.841}{2\pi a\sqrt{\mu\varepsilon}}$$

First higher order mode

TE Modes $f_{c_{nm}} = \dfrac{p'_{nm}}{2\pi a\sqrt{\mu\varepsilon}}$

TM Modes $f_{c_{nm}} = \dfrac{p_{nm}}{2\pi a\sqrt{\mu\varepsilon}}$

n	p'n1	p'n2	p'n3
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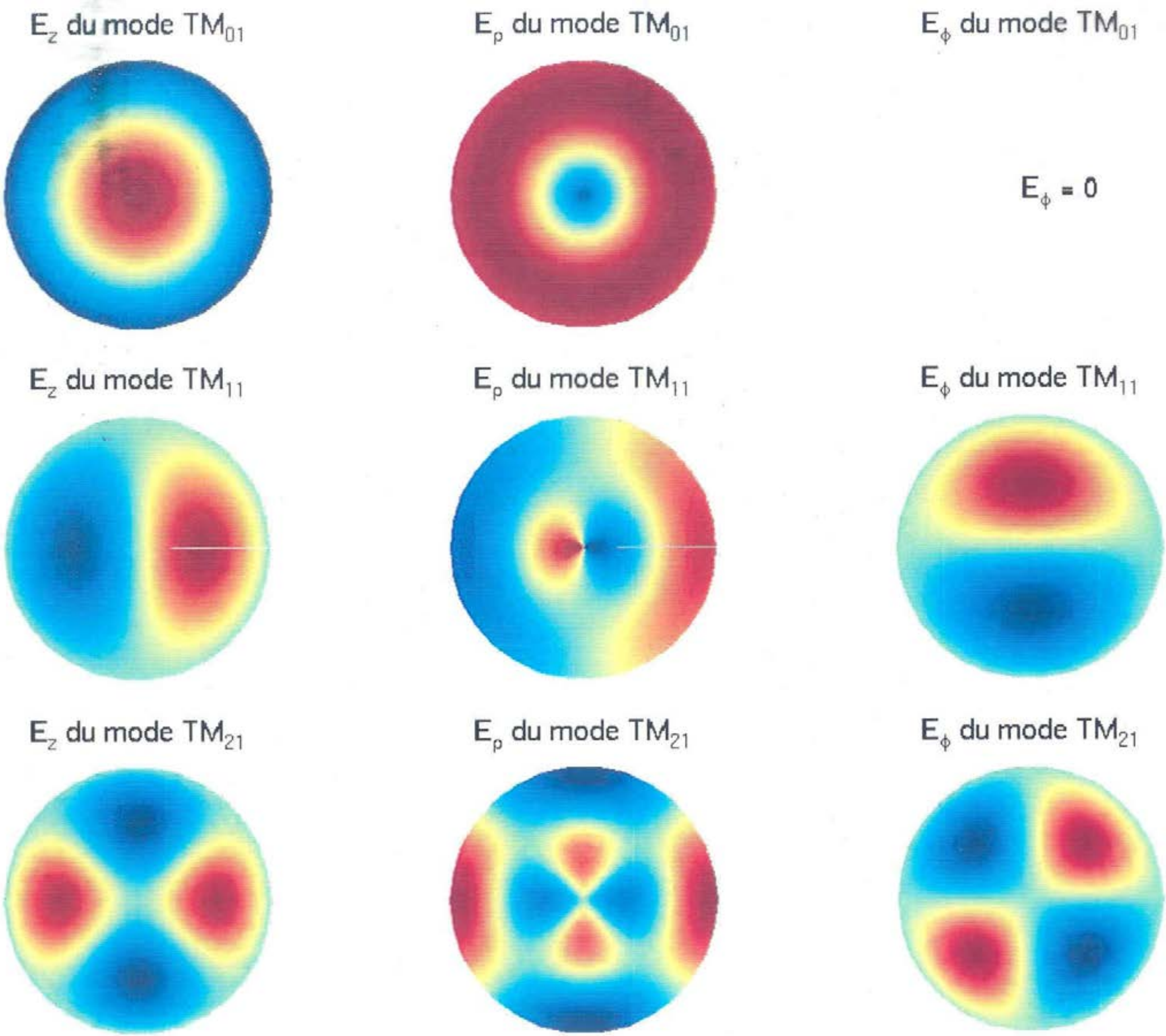
5	8.77142	12.33860	15.70017
6	9.93611	13.58929	17.0038

First higher order mode

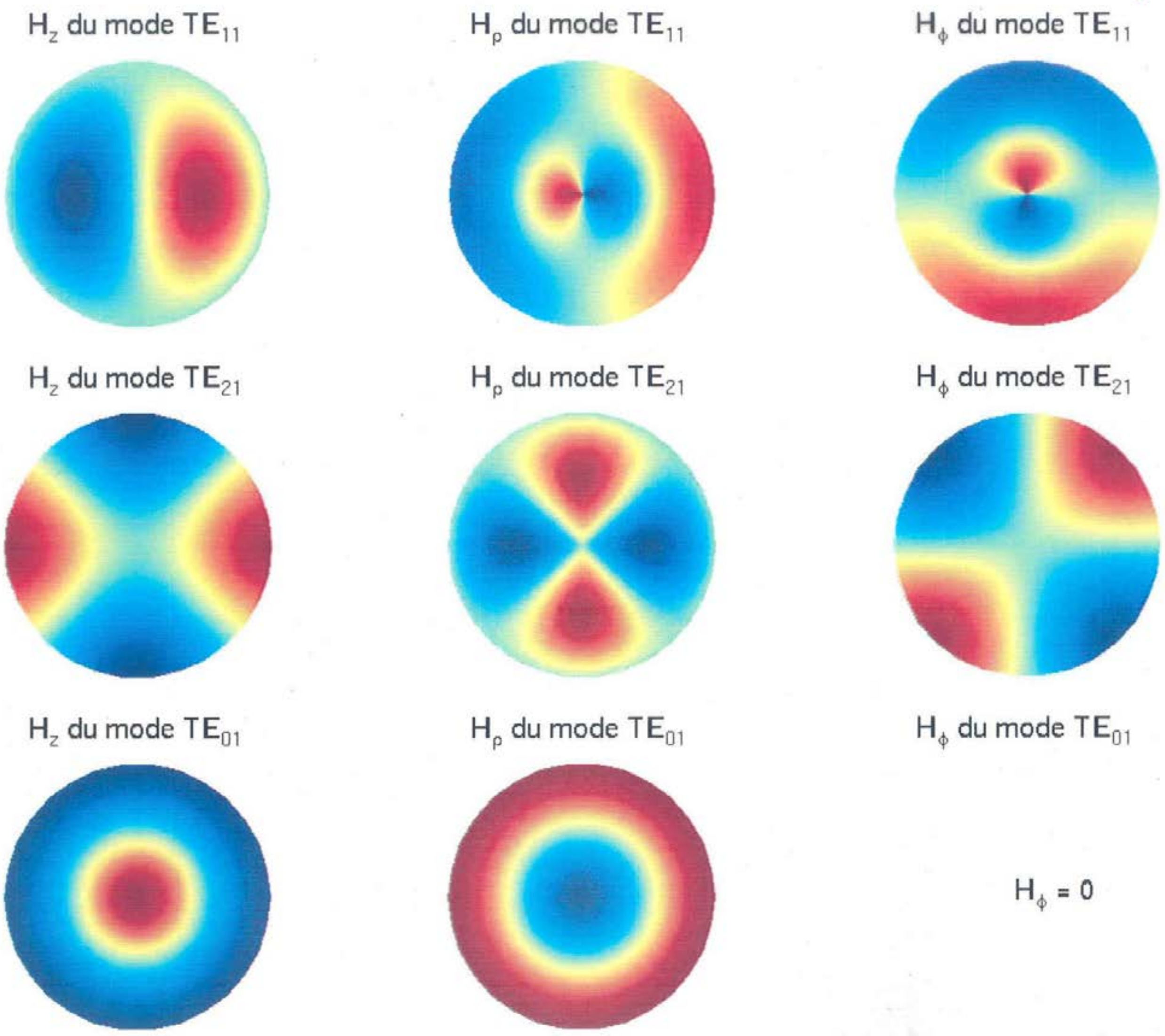
TM_{01} mode

$$f_{c_{nm}} = \frac{2.405}{2\pi a\sqrt{\mu\varepsilon}}$$

Fields in TM Modes



Fields in TE Modes



longitudinal propagation constant

A propagation constant corresponds to each eigenvalue :

$$\gamma = \sqrt{k_c^2 - k^2} \qquad k = \omega\sqrt{\varepsilon\mu}$$

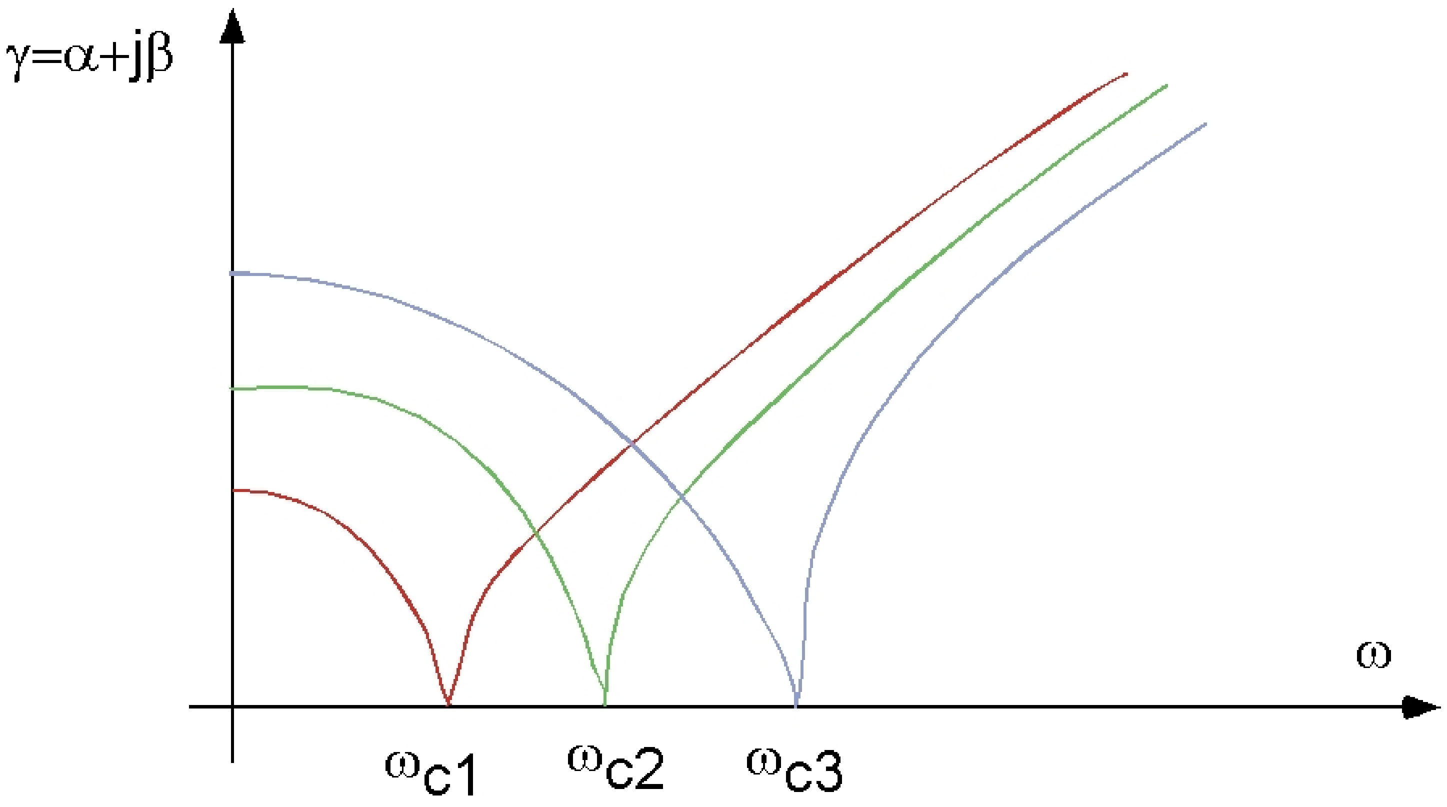
Thus, in a lossless case, depending on the relation between k and k_c the longitudinal propagation constant can be real or imaginary :

$$\text{if } k_c > \omega\sqrt{\varepsilon\mu} \text{ then } \gamma = \alpha$$

$$\text{if } k_c = \omega\sqrt{\varepsilon\mu} \text{ then } \gamma = 0$$

$$\text{if } k_c < \omega\sqrt{\varepsilon\mu} \text{ then } \gamma = j\beta$$

Dispersion diagram



Longitudinal propagation coefficient

$$\gamma = \alpha + j\beta = k_c\sqrt{1-\left(\frac{\omega}{\omega_c}\right)^2} = \frac{\omega_c}{c}\sqrt{1-\left(\frac{\omega}{\omega_c}\right)^2}$$

Group velocity

$$v_g = \left(\frac{\partial\beta}{\partial\omega}\right)^{-1} = c\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}$$

phase velocity

$$v_{\varphi} = \frac{\omega}{\beta} = \frac{c}{\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}}$$

guided wavelength

$$\lambda_g = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{k^2 - k_c^2}} = \frac{\lambda}{\sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^2}}$$

with

$$\lambda_c = \frac{2\pi}{k_c}$$